TUTORIAL

UNIT-WISE QUESTION BANK

ASSIGNMENT RECORD BOOK

GRAPH THEORY

Course Code:	**101903/MA400B**
Branch:	**CSC, IT, AD**
Semester:	**IV**
Academic Year:	**2024-2025**
University:	**Rajagiri School of Engineering &Technology (Autonomous)**

Name of the Student: …………………………………………………….

Reg. No: ………………………… Branch: …………………………..

Faculty in charge: …………………………………………………..

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Name :	**Roll No:**
Branch:	

<u>Module I</u> Introduction to Graphs

Tutorial Questions

Qn. No:	**1**	**2**	**3**	**4**	**5**
Remarks					

Assignment Questions **Date of submission: 01 March 2024**

Qn. No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Remarks											
Marks											

<u>Module II</u> Eulerian and Hamiltonian graphs

Tutorial Questions

Qn. No:	**1**	**2**	**3**	**4**	**5**
Remarks					

Assignment Questions **Date of submission: 15 March 2024**

Qn. No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
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<u>Module III</u> Trees and Graph Algorithms

Tutorial Questions

Qn. No:	**1**	**2**	**3**	**4**	**5**
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Assignment Questions **Date of submission: 12 April 2024**

Qn. No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
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<u>Module IV</u> Connectivity and Planar Graphs

Tutorial Questions

Qn. No:	**1**	**2**	**3**	**4**	**5**
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Qn. No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Remarks											
Marks											

<u>Module V</u> Graph Representations and Vertex Colouring

Tutorial Questions

Qn. No:	**1**	**2**	**3**	**4**	**5**
Remarks					

Assignment Questions **Date of submission: 12 May 2024**

Qn. No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Remarks											
Marks											

Total Marks: ……………….

Signature of the faculty: ……………………….

Bloom's Taxonomy with different difficulty levels

Unit wise Question Bank	Question Number	Difficulty Level
Module 1	1-5	B1, A3, A2, A1, B3
	6-10	B2, B2, A3, C2, B3
	11-15	A2, C3, C1, D1, C3
	16-20	C1, A3, B2, B2, A3
Module 2	1-5	C2, B2, B1, A2, B3
	6-10	B2, C3, C1, B1, C2
	11-15	A1, B2, B3, C1, C2
	16-20	C2, B3, A1, C3, C2
Module 3	1-5	C1, B3, C3, B2, C2
	6-10	A2, C3, A1, C2, B3
	11-15	D2, C3, B1, C3, C2
	16-20	A2, B3, D1, C3, C2
Module 4	1-5	B1, B2, C1, C2, B3
	6-10	C2, C3, B1, D1, A2
	11-15	A1, A3, B3, C2, C3
	16-20	A2, A3, B1, C3, C2
Module 5	1-5	A2, A1, B1, B2, B3
	6-10	A2, A3, D1, B3, C2
	11-15	C1, B3, A3, B2, C2
	16-20	C2, B3, B1, C3, C2

			DIFFICULTY LEVEL		
			1	2	3
			LOW	MEDIUM	HIGH
Learning Objectives	A	Remember	A1	A2	A3
	B	Understand	B1	B2	B3
	C	Apply	C1	C2	C3
	D	Analyze	D1	D2	D3

Module 1- Introduction to Graph Theory

Ready Reckoner

What is a Graph?

A graph G = (V, E) consists of a set of objects $V = \{v_1, v_2, v_3, \ldots.\}$ called vertices and another set $E = \{e_1, e_2 \ldots.\}$, whose elements are called edges.

Self-loop, Parallel Edges and Simple graph

1. A self-loop (or simply loop) is an edge whose endpoints are equal.

2. Multiple edges (or parallel edges) are edges having the same pair of endpoints.

3. A simple graph is a graph having no loops or multiple edges.

Finite and Infinite Graphs

A graph with a finite number of vertices as well as a finite number of edges is called a finite graph; otherwise, it is an infinite graph.

Incidence and Degree

- When a vertex v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with each other.
- Two non-parallel edges are said to be adjacent if they are incident on a common vertex.
- Two vertices are said to be adjacent if they are the end vertices of the same edge.

First Theorem of Graph Theory

In any graph, the sum of all the vertex-degree is equal to twice the number of edges. (also called Handshaking Lemma).

Theorem: There are always even number of odd vertices in a graph.

Regular Graph

- A simple graph in which all vertices are of equal degree is called a Regular Graph.
- If degree of each vertex of a simple graph is k, it is called a k-regular graph.

Null Graph

If in a graph G = (V, E) if the edge set E is empty, is called a Null Graph.

Complete Graph

A complete graph is a simple graph in which every pair of distinct vertices is connected by an edge. A complete graph of "n" vertices is denoted as k_n.

Bipartite Graph:

Let $G = (V, E)$ be a graph, if the vertex set "V" can be partitioned into 2 non empty subsets X and Y (i.e. $V = X \, U \, Y$ & $X \cap Y = f$) in such a way that each edge of G has one end in

"X" and other end in "Y". Then "G" is called a bipartite graph. The partition $V = X \cup Y$ is called a bipartition of G.

Complete Bipartite Graph

A complete Bipartite graph is a simple bipartite graph "G" with bipartition $V = XUY$ in which every vertex in "X" is joined to every vertex of "Y". If "X" has "m" vertices and "Y" has "n" vertices such a graph is denoted by $K_{m,n}$and it has "mXn" edges.

Isomorphism:

Two Graphs G and G' are set to be isomorphic (to each other), if there is a one-to-one correspondence between their vertices and their edges such that the incidence relationship is preserved.

Subgraphs:

A graph "g" is said to a subgraph of G if all the vertices and all the edges of "g" are in G and each edge of "g" has the same end vertices in "g" as in G. It is denoted as g ⊂ G, is stated as "g" is a subgraph of G.

Edge – disjoint subgraph:

- Two (or more) subgraphs g_1 and g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have any edges in common.
- Subgraphs that do not even have vertices in common are said to be **vertex disjoint**.

Walk:

- A walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with vertices preceding and following it. (Vertex may appear more than once).
- It is possible for a walk to begin and end at the same vertex, such a walk is called a closed walk. A walk that is not closed (i.e., the terminal vertices are distinct) is called an open walk.
- A walk is also referred to as an edge train or a chain.
- Vertices with which a walk begins, and ends are called its terminal vertices

Path:

- An open walk in which no vertex appears more than once is called a path OR a simple path OR an elementary path.
- The number of edges in a path is called the length of the path.

Circuit:

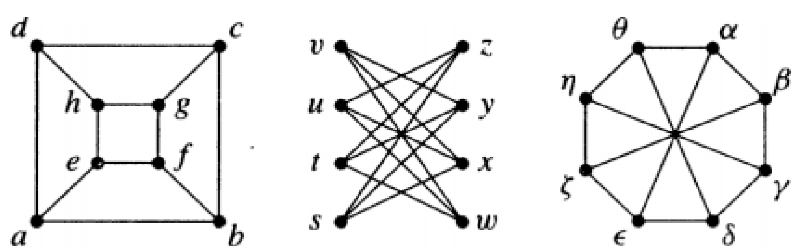
A closed walk in which no vertex appear more than once is called a circuit. (Except the initial and final vertex). A circuit is also called a cycle, elementary cycle, circular path, and polygon.

TUTORIAL QUESTIONS

1. Let G be a graph with n vertices, t of which have degree k and the others have degree $k+1$. Prove that $t = (k+1)n - 2e$.
2. Let G be k-regular graph where k is an odd number. Prove that the number of edges in G is a multiple of k.
3. Draw a graph with four edges, four vertices having degree 1, 2, 3, 4. If not explain why no such graph exists.
4. What is the smallest number n such that the complete graph Kn has at least 500 edges?
5. Can a simple graph exist with 15 vertices, with each of degree 5? Justify your answer.

ASSIGNMENT QUESTIONS

1. Draw a graph on 6 vertices with degree sequence (3, 3, 5, 5, 5, 5), does there exist a simple graph with these degrees? If yes, draw the graph, otherwise, give reason. How are your answers changed if the degree sequence is (2, 3, 3, 4, 5, 5)?
2. Prove that a connected graph G remain connected after removing an edge e from G, if and only if e is in some circuit.
3. Determine which pairs of graphs below are isomorphic and how?

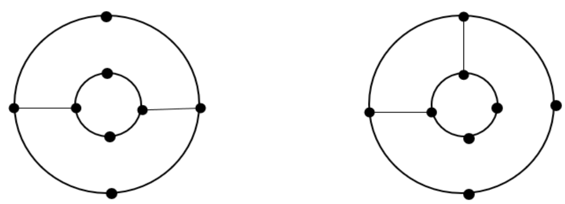


4. Prove that a graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.
5. Draw a simple graph with 8 vertices, 4 components and maximum number of edges.
6. Let G be a graph with n vertices and exactly $n-1$ edges. Prove that G has either a vertex of degree 1 or an isolated vertex.
7. Prove that the number of odd vertices in a graph G is always even. Explain with example.
8. Prove that there is no simple graph on 4 vertices, 3 of which have degree 3 and the remaining vertex has degree 1.
9. Let G be a simple regular graph with n vertices and 24 edges. Find all possible values of n and give examples of G in each case.
10. Let G be a k −regular graph, where k is an odd number. Prove that the number of edges in G is a multiple of k.

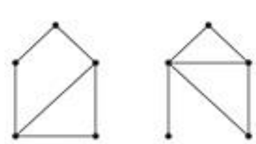
UNIT WISE QUESTION BANK

1. Prove that in a connected graph every $u - v$ walk contains a $u - v$ path.
2. Prepare a short note on walk, trial, path, and cycle of a connected graph.
3. Draw all simple graph of one, two, three, and for vertices.
4. Draw the graph of the following chemical compounds CH_4, C_2H_6, C_6H_6

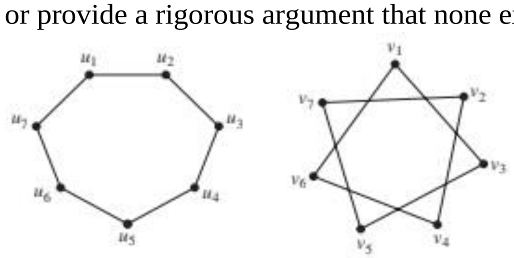
5. Show that the maximum number of edges in a simple graph with ‘n’ vertices is $n(n-1)/2$.
6. Prove that the maximum number of edges in a simple graph with ‘n’ vertices is $(n-1)$.
7. Draw a connected graph that becomes disconnected when any edge is removed from it.
8. Prove that a simple graph with “n” vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.
9. Are the two graphs in the figure isomorphic.



10. Prove that any two simple connected graphs with ‘n’ vertices all of degree 2 are isomorphic.
11. What is the maximum number of edges in a simple graph with ‘n’ vertices?
12. There are 25 telephones in Metropolis. Is it possible to connect them with wires so that each telephone is connected with exactly 7 others? Why?
13. Define complete graph and complete bipartite graph. Draw a graph which is a complete graph as well as a bipartite graph.
14. Explain walk, path, and circuit with the help of examples.
15. Define isolated vertex, pendent vertex, even vertex and odd vertex. Draw a graph that contains all the above.
16. What is the number of vertices in an undirected connected graph with 27 edges,6 vertices of degree 2,3 vertices of degree 4, and the remaining of degree3?
17. Determine whether the graphs are isomorphic. If Yes or No justify



18. Determine whether the given pairs of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



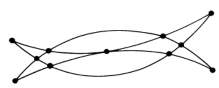
19. Explain seating problem in graph theory using graphical representation.
20. Can five houses be connected to two utilities without connections crossing? Justify your answer by drawing a graph.

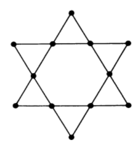
Module 2- Eulerian and Hamiltonian graphs

Ready Reckoner

What is an Euler graph?

If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph an Euler graph .





Theorem:

A given connected Graph G is an Euler graph iff all vertices of G are of even degree.

What is a Unicursal graph?

An open walk that includes all edges of a graph without retracing any edge is called a unicursal line or an open Euler line. A connected graph that has a unicursal line is called a unicursal graph.

Theorem:

In a connected graph G with exactly $2k$ odd vertices, there exists k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

Operations on Graphs:

- Union of Graphs: The union of two Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is another graph G_3=$G_1 \cup G_2$whose vertex set V_3=$V_1 \cup V_2$ and the edge set E_3=$E_1 \cup E_2$.
- Intersection of Graphs: The union of two Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$) is another graph G_3=$G_1 \cap G_2$, whose vertex set V_3=$V_1 \cap V_2$ and the edge set E_3=$E_1 \cap E_2$.
- Ring Sum of graphs: The ring sum of two graphs G_1 and G_2 written as $G_1 \oplus G_2$ is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2.

Hamiltonian path and Hamiltonian Circuit

A Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except the starting vertex at which the walk terminates.

If we remove any one edge from the Hamiltonian circuit, we are left with a path. This is called a Hamiltonian path.

Theorem:

In a complete graph with "n" vertices, there are $\frac{n-1}{2}$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3.

Dirac's Theorem:

A sufficient condition for a simple graph G with $n \geq 3$ vertices have a Hamiltonian circuit is that the degree of every vertex in G be at least $\frac{n}{2}$, where n is the number of vertices in G.

Weighted Graph

In a graph G, with every edge e_i is associated with a real number (the distance in miles or time etc, say) or $w(e_i)$.Such a graph is called a weighted graph; $w(e_i)$being the weight of the edge e_i.

Travelling Salesman problem

In this problem, a travelling salesman wishes to visit several given cities precisely once and return to his starting point, covering the least possible total distance.

Directed Graph

A directed graph or digraph G consists of a set of vertices V=$\{v_1, v_2, ...\}$and a set of edges $E = \{e_1, e_2, ...\}$ and a mapping ψ that maps every edge onto some ordered pair of vertices (v_i, v_j).
In the case of directed graph a vertex is represented by a point and edge by a line segment between v_i and v_j with an arrow directed from v_i to v_j.

Degree of a directed graph:

The number of edges incident out of a vertex v is called the out-degree (or out-valence or outward demi degree) of v and is written $d_+(v)$. The number of edges incident into v is called the in-degree (or in-valence or inward demidegree) of v and is written as $d_-(v)$.

Types of digraphs

- Simple digraph: A digraph that has no loop or parallel edges is called a simple digraph.
- Asymmetric digraph: Digraphs that have utmost one directed edge between a pair of vertices but are allowed to have self-loop are called asymmetric or anti-symmetric digraph.
- Symmetric diagraphs: Digraphs in which for every pair of vertices (a,b) there are edges from a to b and b to a.
- Complete digraph: A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.
  A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

Fleury's algorithm

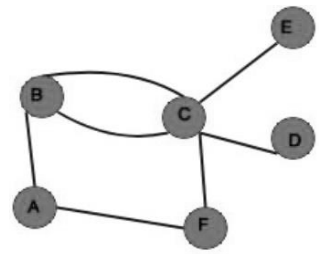
Fleury's Algorithm is used to display the Euler circuit in any given connected graph in which each vertex has even degree. In this algorithm, starting from one edge, it tries to move other adjacent vertices by removing the previous vertices. Using this trick, the graph becomes simpler in each step to find the Euler path or circuit.

Algorithm:

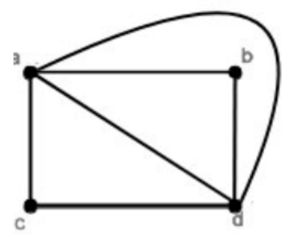
- **Step I** : Start at any vertex. Go along any edge from this vertex to another vertex. Remove this edge from the graph.
- **Step II** : You are now on a vertex on the revised graph. Choose any edge from this vertex, but not a cut edge, unless you have no other option. Go along your chosen edge. Remove this edge from the graph.
- **Step III** : At step II until you have used all the edges and get back to the vertex at which you started.

TUTORIAL QUESTIONS

1. Does the following graph have a Hamiltonian Circuit?

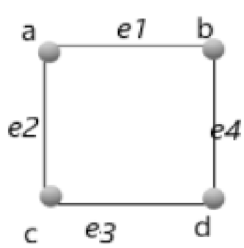


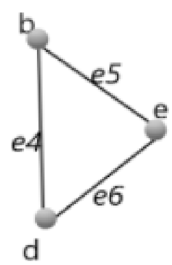
2. Check whether the graph is Eulerian graph. Justify your answer.



3. Can a graph be both Hamiltonian and Eulerian? Justify.
4. Is every Eulerian graph is Hamiltonian give your answer with proper explanation and two examples how they are different from each other on the basis of their definition?

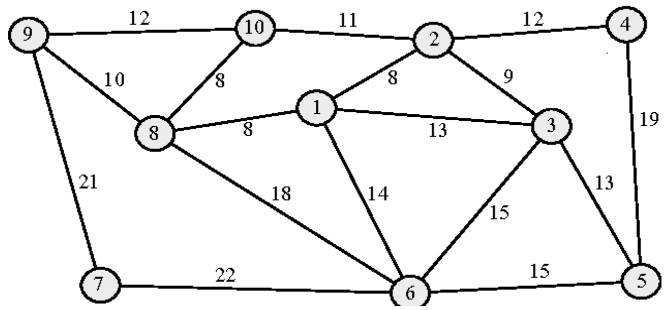
5. Find the union, intersection and ring sum of the following graphs.



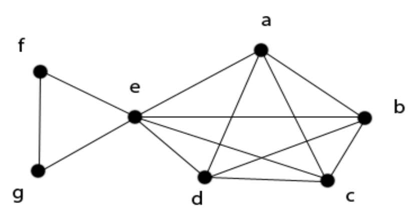


ASSIGNMENT QUESTIONS

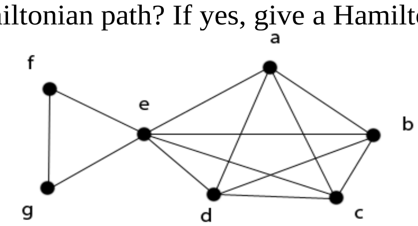
1. Prove that, a connected graph is a Euler graph if and only if it can be decomposed into circuits.
2. Find the number of edge-disjoint Hamiltonian circuits in the complete graph with 9 vertices. Draw all of its edge-disjoint Hamiltonian circuits.
3. Show that a Hamilton path of a graph, if exists, is the longest path in G.
4. State travelling salesman man problem. Give a traveling salesman tour and the distance he travelled on the graph below.



5. Check whether the given graph is Euler graph and if yes, give the Euler walk. Justify your answer.



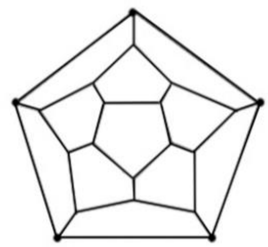
6. Does this graph has Hamiltonian path? If yes, give a Hamiltonian path.



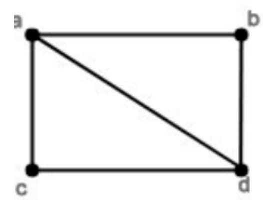
7. Prove that in any digraph the sum of the in-degrees of all vertices is equal to the sum of their out-degrees; and this sum is equal to the number of edges in the graph.
8. Define the following with appropriate example: (i) Reflexive digraph (ii) Symmetric digraph (iii) Transitive digraph (iv) Equivalence digraph.
9. Prove that a connected graph G is a Euler graph iff all vertices of G are of even degree with an example.
10. Give examples a) Hamiltonian circuit b) Unicursal graph c) Complete graph

UNIT WISE QUESTION BANK

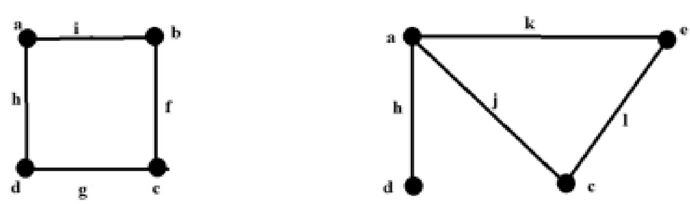
1. Is Hamiltonian circuit always implying Hamiltonian path? Is the converse true? Justify your answer.
2. Is the converse of Dirac's theorem true? Justify your answer.
3. Prove that if a connected graph G is decomposed into two sub graphs g_1 and g_2 there must be at least one vertex common between g_1 and g_2.
4. Is Peterson graph Hamiltonian. Justify your answer.
5. Does the following graph have a Hamiltonian Circuit?



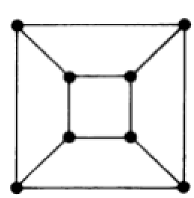
6. Check whether the graph is Eulerian graph. Justify your answer.



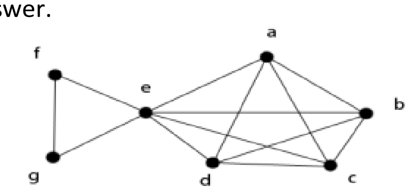
7. Find the union, intersection and ring sum of the following graphs.



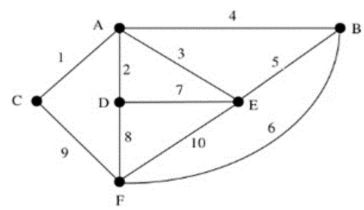
8. State travelling salesman problem. How it is related to Hamiltonian circuits?
9. For which values of m, n is the complete graph k_{mn} an Euler graph ? Justify your answer.
10. State Dirac's theorem and check the applicability of the following graph.



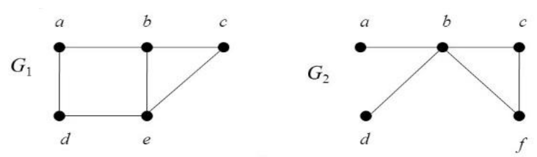
11. Define an Euler graph. Check whether the given graph is Euler graph and if yes, give the Euler walk. Justify your answer.



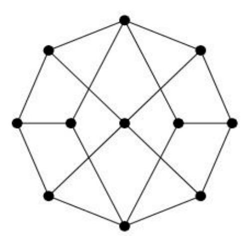
12. Find the smallest integer n such that the complete graph has at least 500 edges.
13. Draw a connected graph that becomes disconnected when any edge is removed is removed from it.
14. Define a Hamilton Cycle. Illustrate. Describe the Travelling Sales man Problem.
15. Fuse the vertices A and B of the following graph



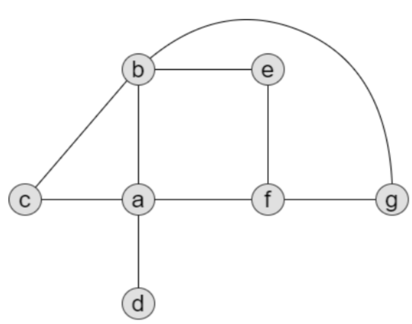
16. Find the union, intersection, and ring sum of the following two graphs



17. For what value of 'n' do the complete graph K_n have an Euler circuit? Give the reason.
18. Consider the following graph, find a Hamiltonian path by labelling the vertices of the given graph, can your path be extended to a Hamiltonian circuit?



19. (a) Using figurative representation and explain the difference between complete symmetric digraph and complete asymmetric digraph. (b) When are two digraphs considered to be the same or isomorphic, explain with example.
20. Use Fleury's Algorithm to find Euler path or Euler Circuit in the following graph

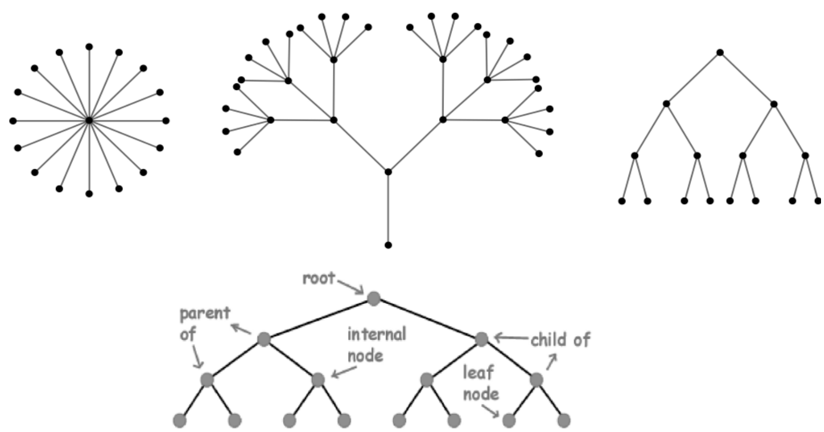


Module 3- Trees and Graph Algorithms

Ready Reckoner

<u>Tree</u>

Tree is a simple connected acyclic graph.



<u>Properties of Trees:</u>

- There is one and only one path between every pair of vertices in a tree T.
- If in a graph G there is one and only path between every pair of vertices, G in a tree.
- A tree with "n" vertices has $(n-1)$ edges.
- Any connected graph with "n" vertices and $n-1$ edges is a tree.
- A graph G with "n" vertices, $(n-1)$ edges, and no circuit is connected.
- In any tree (with two or more vertices) there are at least two pendant vertices.

<u>Minimally Connected</u>

A connected graph G is said to be minimally connected if removal of any one edge from it disconnects the graph.

<u>Theorem:</u>

A graph is a tree if and only if it is minimally connected.

<u>Distance of a Tree</u>

In a connected graph G, the distance d(u,v) between two of its vertices u and v is the length of the shortest path (that is the number of edges in the shortest path) between them.

<u>Theorem</u>

The distance between the vertices of a connected graph is a metric.

<u>Eccentricity</u>

The eccentricity $E(v)$ of a vertex 'v' in a graph G is the distance from v to the vertex farthest from v in G. It is denoted as $E(v) = \underset{v_i \in G}{Max} d(v, v_i)$

<u>Centre:</u>

A vertex with minimum eccentricity in a graph G is called the center of G.

Theorem

Every tree has either one or two centres

Binary Tree

A binary tree is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices are of degree one or three.

Cayley's Theorem

The number of labelled trees with "n" vertices $(n \geq 2)$is n^{n-2}

Spanning Tree

A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

Kruskal's Algorithm

Given a weighted graph "G",

- Step 1: Choose a link (edge) e_1 such that $w(e_1)$ is as small as possible
- Step 2: if edges $e_1, e_2, \cdots, e_i$ have been chosen, then an edge e_{i+1} from $E - \{e_1, e_2, \cdots, e_i\}$ in such a way that $G[\{e_1, e_2, \cdots, e_{i+1}\}]$ is acyclic $w(e_{i+1})$ is as small as possible subject to (i).
- Step 3: Stop when Step 2 cannot be implemented further.

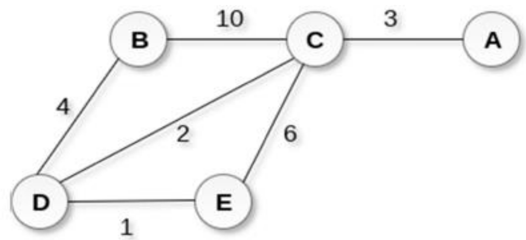
Floyd's Algorithm

- Step 1 : set k = 0
- Step 2: form the initial distance matrix $[D^0]$ and the initial precedence matrix $[P^0]$. Form the network.
- Step 3: set k = k+1
- Step 4: Obtain the values of the distance matrix D^k for all the cells where $i \neq j$ using the following formula

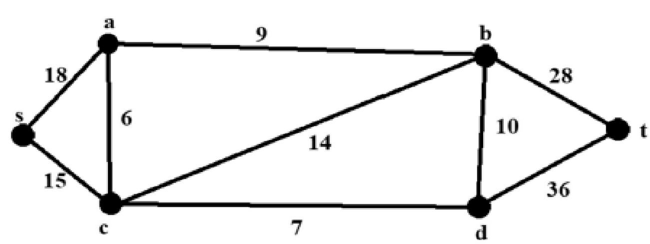
$$D_{ij}^k = min\left[D_{ij}^{k-1}, D_{ik}^{k-1} + D_{kj}^{k-1}\right]$$

TUTORIAL QUESTIONS

1. How many labelled trees are there with n vertices? Draw all labelled trees with 3 vertices.
2. Prove that a connected graph G with n vertices and n-1 edges is a tree.
3. Prove that a binary tree with n vertices has (n+1)/2 pendant vertices.
4. Using Prims algorithm, find a minimal spanning tree for the following graph.

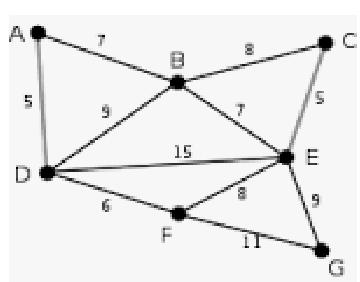


5. Write down Dijkstra's algorithm and use it to find the shortest path from s to t

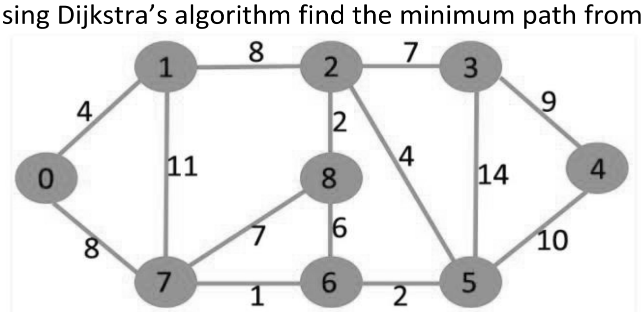


ASSIGNMENT QUESTIONS

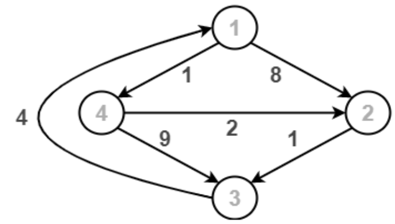
1. Draw all non-isomorphic trees with six vertices.
2. Using Kruskal's algorithm find the minimum spanning tree



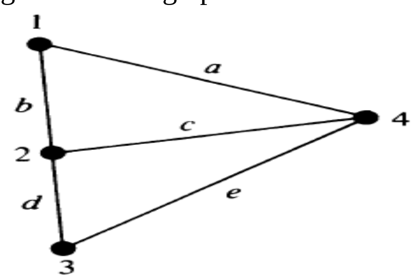
3. Using Dijkstra's algorithm find the minimum path from vertex 0 to 4.



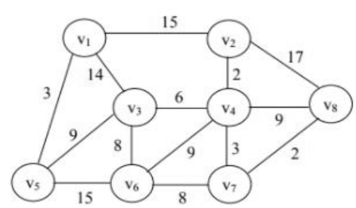
4. Using Floyd Warshall Algorithm find the shortest path between every pair of vertices.



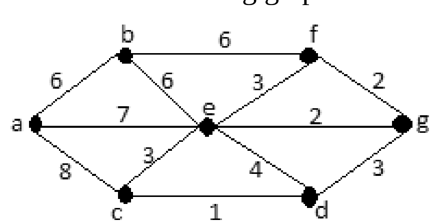
5. Sketch all spanning trees of the graph



6. Show a tree in which its diameter is not equal to twice the radius.
7. Let T be a tree and let v be a vertex f maximum degree in T ,say d(v)=K,prove that T has atleast K vertices f degree 1
8. Prove that any tree with atleast 2 vertices is a bipartite graph.
9. Using Dijkstra's method find the shortest path from node V1 to node V8 from the following network path model.

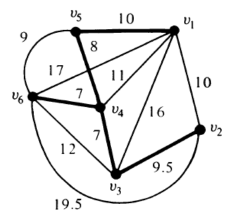


10. Write Kruskal's algorithm for finding out an optimal tree in a graph and apply it to find out the optimal tree for the following graph

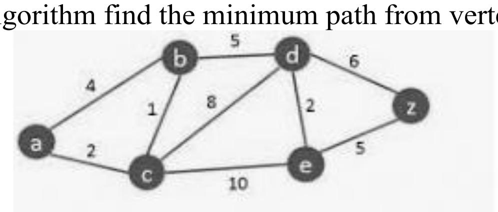


UNIT WISE QUESTION BANK

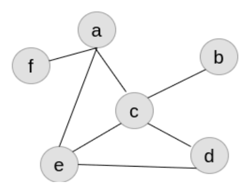
1. Draw all non-isomorphic trees with seven vertices.
2. What is the nullity of a complete graph with n vertices?
3. Show that Hamiltonian path is a spanning tree.
4. Using Kruskal's algorithm find the minimum spanning tree



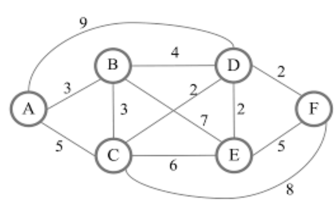
5. Using Dijkstra's algorithm find the minimum path from vertex a to z.



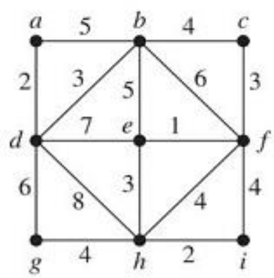
6. Find the Eccentricity, Diameter, Radius and Centre of the graph shown below:



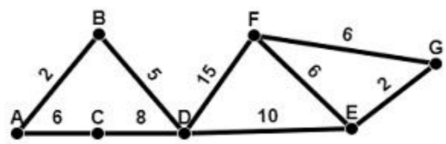
7. Prove that every connected graph has at least one spanning tree.
8. Explain Dijkstras Algorithm and find the shortest path from A to F.



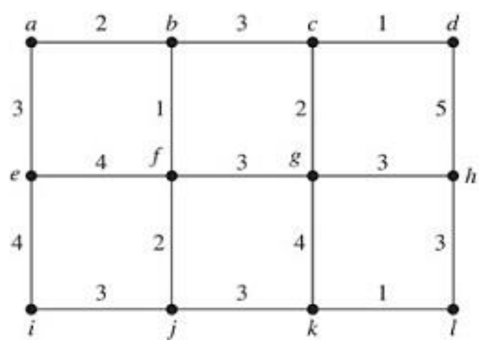
9. Prove that a tree with n vertices has $n-1$ edges.
10. Prove that a pendant edge in a connected graph G is contained in every spanning tree of G
11. Cite three different situations that can be represented by trees. Explain.
12. Prove that any circuit in a graph G must have atleast one edge in common with a chord set.
13. How many isomers does Pentane C_5H_{12} have?
14. Prove that two colours are necessary and sufficient to paint all n vertices $(n \geq 2)$ of a tree, such that no edge in the tree has both of its end vertices of the same colour
15. Find the maximum and minimum level of a rooted binary tree with 75 vertices
16. Use Prim's algorithm to find a minimum spanning tree of the weighed graph given below



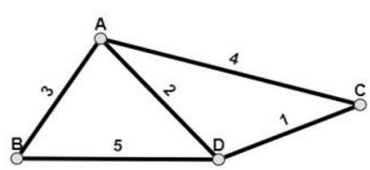
17. Describe Dijkstra's algorithm and find the shortest path from a to z in the following graph.



18. Describe Kruskal's algorithm and apply it to find the minimal spanning tree for the following graph.



19. Describe Floyd Warshall algorithm and use it to find shortest distance between all pairs of vertices in the following graph



20. Define branches and chords of a graph. Find the number of chords in a graph with 17 vertices and 24 edges.

Module 4- Connectivity and Planar Graphs

Ready Reckoner

Vertex Connectivity

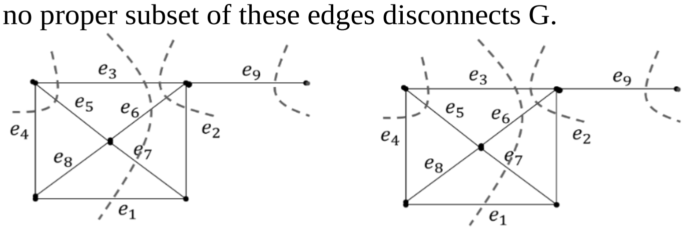
The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the graph disconnected. Vertex connectivity of a tree is 1.

Edge Connectivity

The edge connectivity of a connected graph can be defined as the minimum number of edges whose removal (deletion) reduces the rank of the graph by 1.The edge connectivity of a tree is 1

Cut-Set

In a connected graph G, a cut-set is a set of edges whose removal from G leaves G disconnected, provided removal of no proper subset of these edges disconnects G.



$\{e_2, e_3, e_6, e_7\}$is not a cut-set because one of its proper subsets $\{e_1, e_3, e_6\}$ is a cut-set

The Edge set $\{e_1, e_3, e_6, e_7\}$ $\{e_3, e_4, e_8\}$ is a cut-set. Also $\{e_9\}$ is a cut-set.

Theorem: Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G.

Theorem: Every circuit has an even number of edges in common with any cut-set.

Fundamental circuits and cut-set:

- A cut-set S contains exactly one branch of a spanning tree T of a given connected graph G is called a fundamental cut-set with respect to T. A fundamental cut set is also called a basic cut set.
- Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T, and let the fundamental circuit made by c_i be called Γ_i , consisting of k branches $b_1, b_2, \dots, b_k$ in addition to the chord c_i that is $\Gamma_i = \{c_i, b_1, b_2, \dots, b_k\}$ is a fundamental circuit with respect to T.

Theorem : With respect to a given spanning tree 'T', a chord c_i that determines a fundamental circuit Γ_i, occurs in every fundamental cut set associated with the branches in Γ_i and in no other.

Theorem: With respect to a given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chord in S, and in no others.

- In the previous example, branch $b_3$of the spanning tree $\{b_1, b_2, b_3, b_4, b_5\}$. The fundamental cut set determined by b_3 is $\{b_3, c_2, c_4\}$.
- The two fundamental circuits determined by chords c_2 and c_4 are
- $\Gamma_2 = \{c_2, b_2, b_3\}$ and $\Gamma_4 = \{c_4, b_3, b_4, b_5\}$

- Branch b_3 is contained in both these fundamental circuits and none of the remaining 3 fundamental circuits contain the branch b_3.

Theorem: If $G\ (V, E)$ is a connected graph then v is a cut vertex iff there exists vertices $u,\ w\ \in V - \{v\}$ such that every $u - w$ path in G passes through v.

Vertex Connectivity (or simply connectivity)

Vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected, denoted by $K(G)$.

Separable graph

A connected graph is said to be separable if its vertex connectivity is one.

Cut-Vertex

In a separable graph, a vertex whose removal disconnects the graph is called a cut-vertex or a cut-node or an articulation point.

Connectivity and Separability

Each cut-set of a connected graph G consists of a certain number of edges. The number of edges in the smallest cut-set (i.e. cut-set with fewest number of edges) is defined as the edge connectivity of G. Denoted by $\lambda(G)$.

Theorem-I: The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Theorem-II: The vertex connectivity of any graph G can never exceed the edge connectivity of G. $(K(G) \leq \lambda(G) \leq \delta(G))$

Theorem-III:

The maximum vertex connectivity one can achieve with a graph G of "n" vertices and "e" edges $(e \geq n-1)$ is $\left\lfloor \frac{2e}{n} \right\rfloor$ (Integral part of the number $\frac{2e}{n}$)

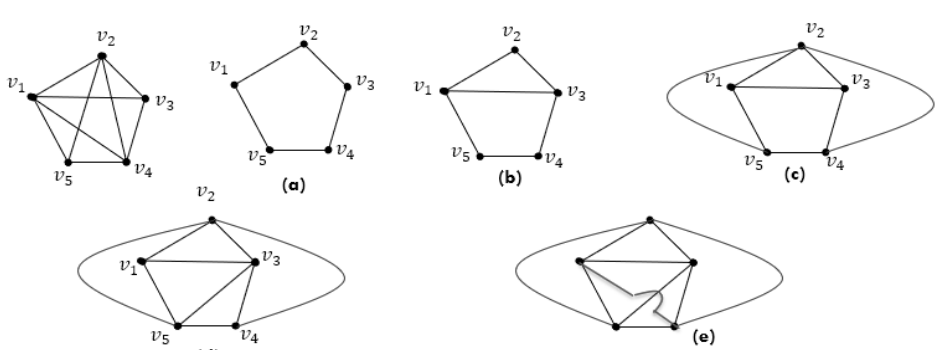
Planar Graph

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect.

- Note: To prove that a graph is planar, if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is non-planar.
- An embedding of a planar graph G on a plane is called a plane representation of G

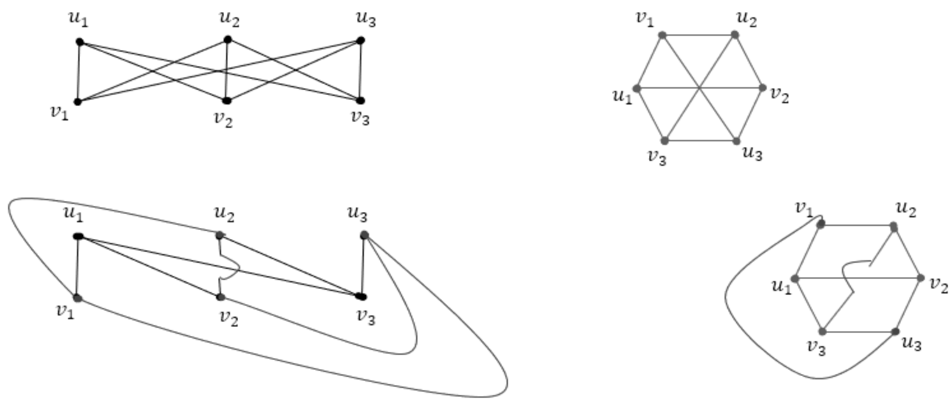
Kuratowski's Two graphs

Theorem: The complete graph of five vertices is non-planar.



Kuratowski's second graph

The second graph of Kuratowski is a regular connected graph with six vertices and 9 edges as shown in the figure (a) and (b) below.



Properties common to the two graphs of Kuratowski

- Both are regular graphs
- Both are non-planar
- Removal of one edge or a vertex makes each a planar graph
- Kuratowski's first graph is the non-planar graph with the smallest number of vertices, and Kuratowski's second graph is the non-planar graph with the smallest number of edges.
- Thus, both are the simplest non-planar graphs.

Different representations of planar graph

Region

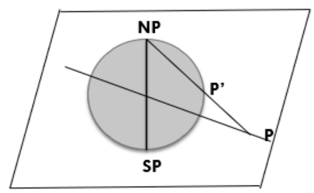
A plane representation of a graph divides the plane into regions (also called windows, faces, or meshes)A Region is characterized by the set of edges (or the set of vertices) forming its boundary.

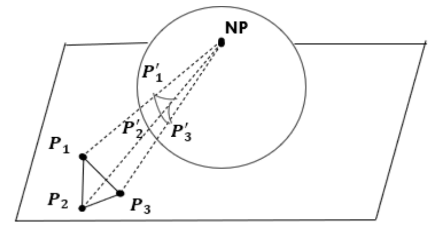
Embedding on a sphere

- To eliminate the distinction between finite and infinite regions, a planar graph is often embedded in the surface of a sphere. It is accomplished by stereographic projection of a sphere on a plane.
- Put the sphere on the plane and call the point of contact SP (South Pole). From the point SP, draw a straight line perpendicular to the plane, and let the point where this line intersect the surface of the sphere be called the NP (North Pole).
- Now corresponding to any point P on the plane, there exists a unique point P` in the point at which the straight line from P to NP intersects the surface of the sphere.

Stereographic Projection

Thus, there is a one-to-one correspondence between the points of the sphere and the finite points on the plane and points at infinity in the plane corresponding to the point NP on the sphere.





Euler's Theorem:

A connected planar graph with n vertices, f faces and e edges satisfies the relation $e - n + 2 = f$ or $f + n - e = 2$.

Corollary:

In any simple, connected planar graph with f region, n vertices, and e edges $(e > 2)$, the following inequalities must hold,

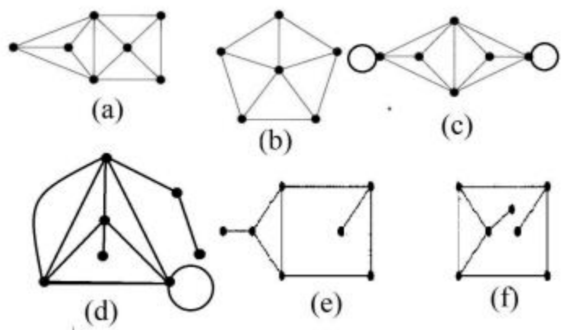
$e \geq \frac{3}{2} f,\ e \leq 3n - 6.$

Dual graph or Geometric dual or Combinatorial dual graph

Given a planar graph, its geometric dual is constructed by placing a vertex in each region (including exterior region) and if two regions have an edge in common, join the corresponding edges.

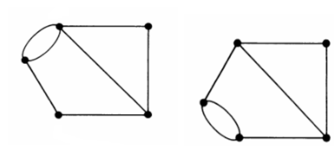
TUTORIAL QUESTIONS

1. Using geometric arguments, prove that Kuratowski's second graph is also nonplanar.
2. Show that the complete graph of four vertices is self-dual. Give another example of a self-dual graph.
3. Let G be a connected planar graph with 12 vertices, 30 edges and degree of each region is k. Find the value of k.
4. Give alternate proof of Euler's formula.
5. Find the dual of the following graph shown in figure

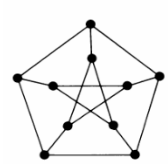


ASSIGNMENT QUESTIONS

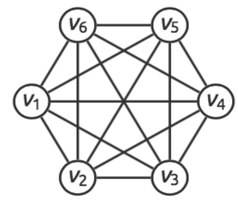
1. Prove that a vertex v of a tree **T** is a cut vertex if and only if $d(v) = 1$.
2. Show that if e is any edge of K_5 then $K_5 - e$ is planar.
3. Show by actual construction that the geometric dual of the two isomorphic graphs in the following figures are isomorphic.



4. Show that if e is any edge of $K_{3,3}$ then $K_{3,3} - e$ is planar.
5. Using Kuratowski's theorem, show that Petersen's graph is non-planar.



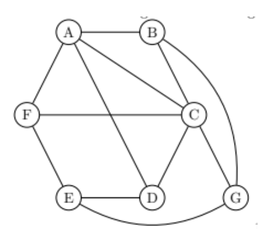
6. How many vertices does a connected planar graph G have if it has 7 edges and can be drawn without any edge crossings with 4 faces?
7. We already know that the complete bipartite graph K3,3 is not planar. How many edges does one need to remove from it to make it planar?
8. Is the complete graph K6 on 6 vertices planar explain?



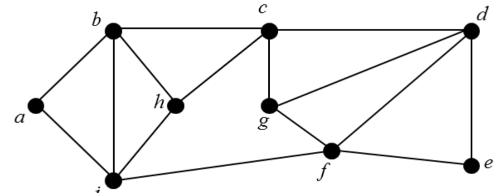
9. Let G be a plane 4-regular graph with 10 faces. Determine how many vertices G has and give a drawing of such a graph.
10. Prove that if G is a simple connected bipartite plane graph with e edges, where $e \geq 3$, and n vertices then $e \leq 2n - 4$.

UNIT WISE QUESTION BANK

1. Define edge connectivity, vertex connectivity and separable graphs. Give an example for each.
2. Prove the statement using an example: Every cut set in a connected graph G must also contain at least one branch of every spanning tree of G.
3. State Kuratowski's theorem and use it to show that the graph G below is not planar. Draw G on the projective plane without edges crossing. Your drawing should use the labelling of the vertices given.

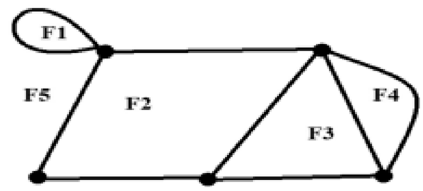


4. Let G be a connected graph and e an edge of G. Show that e is a cut-edge if and only if e belongs to every spanning tree.
5. If G is a 5-regular simple graph and |V| = 10, prove that G is non-planar.
6. State and prove Euler's Theorem relating the number of faces, edges and vertices for a planar graph.
7. From the following graph form a spanning tree and give all the fundamental cut set and fundamental circuits.

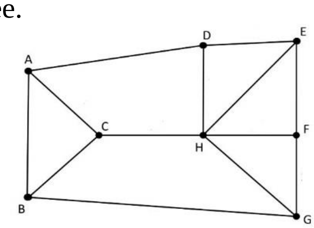


8. Define fundamental circuit and fundamental cut-sets.

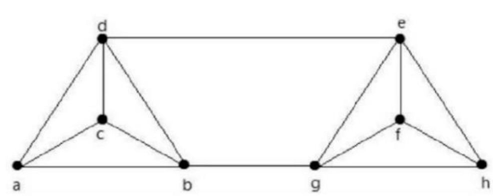
9. Show that a set of fundamental circuits in a planar graph G corresponds to a set of fundamental cut sets in its dual G^*.
10. Define planar graphs. Is K_4, the complete graph with 4 vertices, a planar graph?
11. Define cut-set. Prove that every circuit in G has an even number of edges in common with any cut-set.
12. Construct the geometric dual of the graph given below



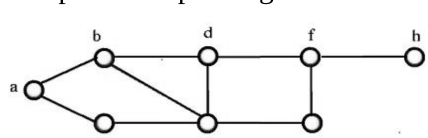
13. Prove that a connected planar graph with n vertices and e edges has $e - n + 2$ regions.
14. Let G be a connected graph and e an edge of G. Show that e is a cut-edge if and only if e belongs to every spanning tree.
15. Prove that in any simple, connected planar graph with f region, n vertices, and e edges $(e > 2)$, the following inequalities must hold $e \geq \frac{3}{2}f$, $e \leq 3n - 6$.
16. Define fundamental cut set of a graph. Find all fundamental cut set of the graph below with respect to a spanning tree.



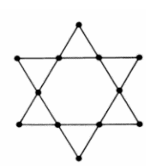
17. Calculate the vertex connectivity edge connectivity and minimum vertex degree of the following graph.



18. How many vertices does a connected planar graph G have if it has 7 edges and can be drawn without any edge crossings with 4 faces?
19. With respect to a spanning tree for the following graph find all fundamental circuits.



20. Draw a geometric dual of the graph



MODULE 5- Graph Representations and Vertex Coloring

Ready Reckoner

Incidence Matrix

Let G be a graph with n vertices, e edges and no self-loops. Define $n \times e$ matrix $A = [a_{ij}]$, whose n rows correspond to the n vertices and the e columns to e edges as follows where

$a_{ij} = 1$, if j^{th} edge e_j is incident on i^{th} vertex v_i, and

= 0, otherwise.

Theorem: If A(G) is an incidence matrix of a connected graph G with "n" vertices, then the rank of A(G) is $(n - 1)$.

Circuit Matrix

Let the number of different circuits in a graph G be q and the number of edges in G be 'e'. Then the circuit matrix $B = [b_{ij}]$of G is a q by e, (0,1) matrix defined as follows

$b_{ij} = 1$, if i^{th} circuit includes j^{th} edge, and

= 0, otherwise.

Theorem: Let B and A be respectively the circuit matrix and the incidence matrix of a self-loop free graph, whose columns are arranged using the same order of edges. Then every row of B is orthogonal to every row of A. That is $\boldsymbol{AB^T = BA^T \equiv 0(mod 2)}$.

Path matrix

A path matrix is defined for a specific pair of vertices in a graph, say (x,y) and is written as P(x,y).The rows in P(x,y) corresponds to different paths between the vertices x and y and the column corresponds to the edges in G i.e. the path matrix for (x,y) vertices is

$p(x, y) = [P_{ij}]$ where $P_{ij} = \begin{cases} 1 & if\ j^{th} edge\ lies\ in\ i^{th} path \\ 0 & otherwise \end{cases}$

Adjacency Matrix

The adjacency matrix of a graph G with "n" vertices and no parallel edges is an $n \times n$ symmetric binary matrix $X = [x_{ij}]$ defined as

$$x_{ij} = \begin{cases} 1, & if\ there\ is\ an\ edge\ between\ i^{th}\ and\ j^{th}\ vertices \\ 0, & if\ there\ is\ no\ edge\ between\ them \end{cases}$$

Vertex coloring

- K – vertex coloring:
  - A k-vertex coloring of a graph G is an assignment of k colors, 1, 2…,k to the vertices of G. The coloring is proper if no two distinct adjacent vertices have the same color.
- K-vertex colourable:
  - A graph G is k-vertex colourable if G has a proper k-vertex coloring. K- vertex colourable is also called as k-colourable.

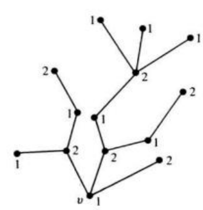
- Chromatic Number $\chi(G) = \kappa$:
  - The chromatic number of a graph G is denoted by $\chi(G) = \kappa$ is the minimum κ for which, G is κ -colourable.

<u>Chromatic Number</u>

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the **PROPER COLORING** of a graph.

A graph G that requires κ different colors for its proper coloring and no less is called a κ -chromatic graph and the number κ is called the <u>chromatic number</u> of G.

Theorem 1: Every tree with two or more vertices is 2-chromatic



Theorem 2: A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

<u>Chromatic Polynomial</u>

- We have seen that a given graph G of n vertices can be properly colored in many ways using a sufficient large number of colors.
- This property of a graph is expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G
- Definition:
  - The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices give the number of ways of properly coloring the graph, using λ or fewer colors.

Theorem:

Given G be a graph with n vertices. Let x and y be two non-adjacent vertices in G. From G, we can compute two graphs G1 and G2 .G1 be a simple graph obtained from G by adding an edge between x and y. G2 be a simple graph obtained from G by fusing the vertices x and y together and replacing the parallel edges with single edge. Then the chromatic polynomial of G

Decomposition of the graph can be done till all the vertices become adjacent.

$P_n(\lambda)$ of G = $P_n(\lambda)$ of G1 + $P_{n-1}(\lambda)$ of G2

Theorem: A graph of n vertices is a complete graph iff its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1) \ldots (\lambda - n + 1)$.

Theorem: An n −vertex graph is a tree iff its chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{(n-1)}$

Matching/Independent edge set

Let G be a graph with no self-loop. A subset M of the edge set E of a graph G is called a matching in G if no two edges in M are adjacent in G.

M-Saturated vertex

Let M be a matching in a graph G. A vertex ‘v’ is said to be M-saturated if there is an edge in M incident with ‘v’. Otherwise ‘v’ is said to be M-unsaturated.

Perfect Matching

A matching M in the graph G is said to be perfect if it saturates every vertex of G.

Coverings

In a graph G, a set E of edges is said to cover G if every vertex in G is incident on at least one edge of E. A set of edges that covers a graph G is said to be an edge covering, a covering subgraph, or simply a covering of G.

Minimal covering

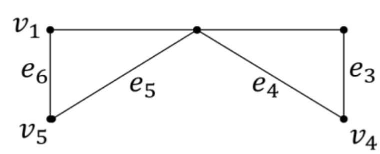
A covering from which no edge can be removed without destroying its ability to cover the graph.

<u>**Tutorial Questions**</u>

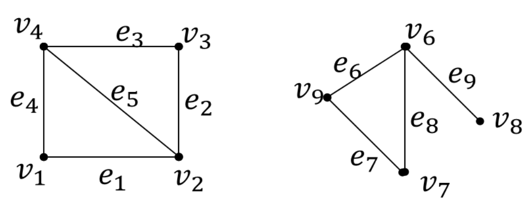
1. Given the incidence matrix of G, draw the graph of G.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

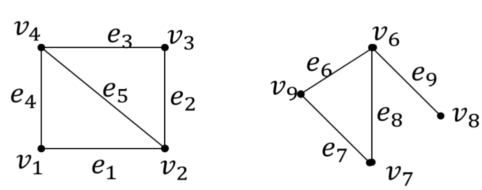
2. Construct the incidence matrix A for the graph shown below



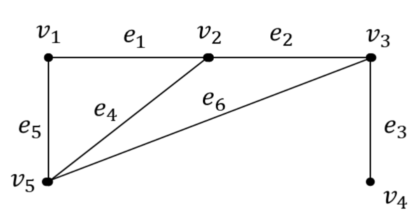
3. Consider the following disconnected graph G with two components. Find the incidence matrix of G.



4. Consider the following disconnected graph G with two components. Find the circuit matrix of G.

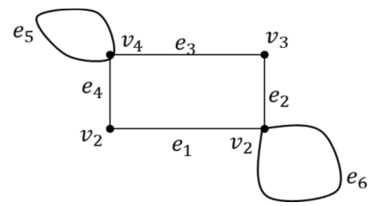


5. Verify $AB^T = BA^T \equiv 0(mod2),$ where B and A be respectively the circuit matrix and the incidence matrix of the given graph

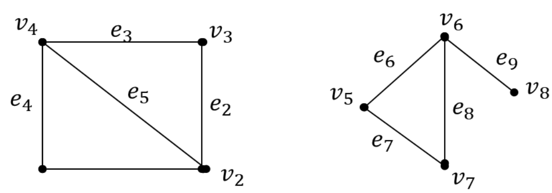


ASSIGNMENT QUESTIONS

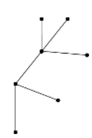
1. Construct an adjacency matrix X for the following graph.



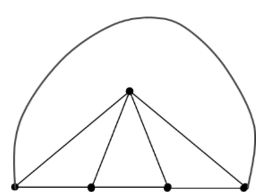
2. Construct an adjacency matrix X for the following graph disconnected graph and write the properties of the adjacency matrix.



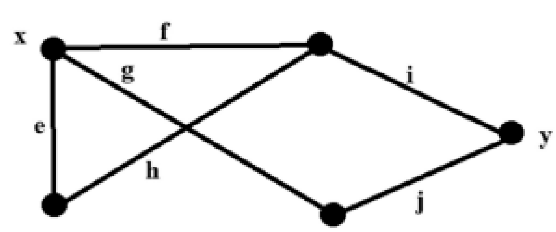
3. Find the chromatic polynomial for the following graph



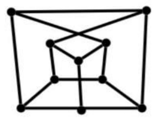
4. Show that the chromatic polynomial of a graph of ‘n’ vertices satisfies the inequality $P_n(\lambda) \leq \lambda(\lambda-1)^{n-1}$.
5. Find the chromatic polynomial of the given graph



6. Define path matrix of a graph. Find the path matrix $P(x, y)$ for the graph below



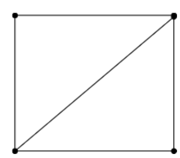
7. What will be the chromatic number for the following graph?



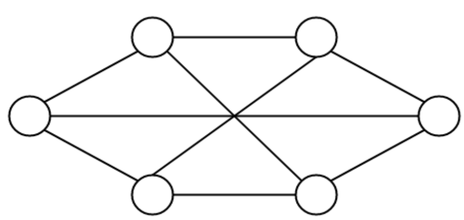
8. Prove that if A(G) is an incidence matrix of a connected graph G with "n" vertices, then the rank of A(G) is $(n - 1)$.
9. Find the chromatic polynomial of a complete graph with 'n' vertices.
10. Show that chromatic polynomial of a tree with n vertices is $P_n(\lambda) = \lambda(\lambda - 1)^{(n-1)}$

UNIT WISE QUESTION BANK

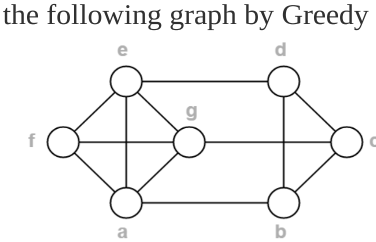
1. Using induction on 'n', prove that an 'n-vertex' is a tree if and only if its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1)^{n-1}$.
2. Sketch two different (non-isomorphic) graphs that have the same chromatic polynomial.
3. Find the chromatic polynomial of the graph



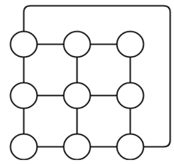
4. Find chromatic number of the following graph



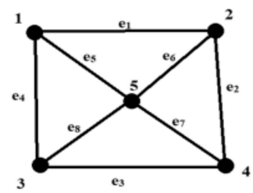
5. Find chromatic number of the following graph by Greedy Algorithm



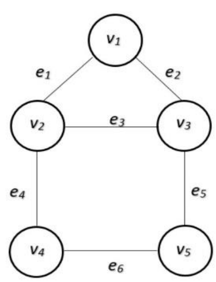
6. Find chromatic number of the following graph



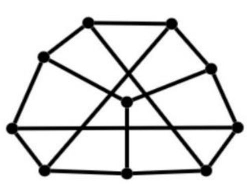
7. Construct the adjacency matrix and incidence matrix of the following graph



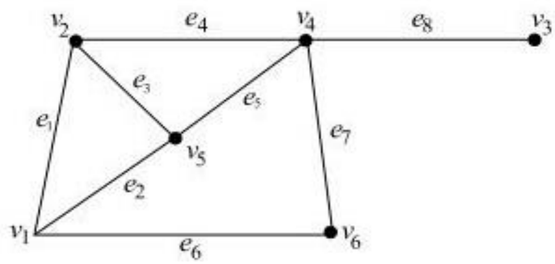
8. Define chromatic number. What is the chromatic number of a tree with two or more vertices?
9. A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.
10. Prove that a graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.
11. Prove that a graph of n vertices is a complete graph iff its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1) \ldots (\lambda - n + 1)$.
12. Explain four colour problem using the concept of chromatic number.
13. Construct incidence matrix for the following graph



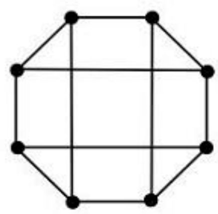
14. What will be the chromatic number for the following graph?



15. Prove that an n −vertex graph is a tree iff its chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{(n-1)}$
16. Construct the adjacency matrix X for the following graph and write properties of adjacency matrix



17. Colour the faces of K_4 with 4 colours. Can only 3 colours be used?
18. What will be the chromatic number for the following graph?



19. Prove that a graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.
20. Let B and A be respectively the circuit matrix and the incidence matrix of a self-loop free graph, whose columns are arranged using the same order of edges. Then show that $AB^T = BA^T \equiv 0(mod 2)$.