RSET
RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY (AUTONOMOUS)

TUTORIAL

UNIT-WISE QUESTION BANK

ASSIGNMENT RECORD BOOK

COMPLEX ANALYSIS AND TRANSFORMS

Course Code: 102903/MA200A
Branch: CS, IT, AD
Semester: II
Academic Year: 2023-2024
University: Rajagiri School of Engineering &Technology (Autonomous)

Name of the Student: ……………………………………………………….

Reg. No: ………………………… **Branch:** ……………………………..

Faculty in charge: ………………………………………………………..

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INDEX

Name:	**Roll No:**
Branch:	

<u>Module I</u> Complex Differentiation

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission: 01 March 2024**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

<u>Module II</u> Complex Integration

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission: 15 March 2024**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
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<u>Module III</u> Laplace transforms

Tutorial Questions

Qn. No:	1	2	3	4	5
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Assignment Questions **Date of submission: 12 April 2024**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
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Module IV Fourier Transforms

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission: 23 April 2024**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

Module V Z Transforms

Tutorial Questions

Qn. No:	1	2	3	4	5
Remarks					

Assignment Questions **Date of submission: 12 May 2024**

Qn. No:	1	2	3	4	5	6	7	8	9	10	Total
Remarks											
Marks											

Total Marks: ……………….

Signature of the faculty: ………………………

Bloom's Taxonomy with different difficulty levels

Unit wise Question Bank	Question Number	Difficulty Level
Module 1	1-5	B1, B2, C2, C1, D2
	6-10	A3, B3, C2, C2, C2
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Module 3	1-5	C2, C2, C2, C3, C3
	6-10	C2, C3, C2, C2, C2
	11-15	C2, C3, B2, B3, D1
	16-20	B2, B3, C1, C3, C2
Module 4	1-5	C2, C1, C3, C2, C2
	6-10	B2, C2, C2, C3, B3
	11-15	C1, A3, B3, B2, C2
	16-20	C2, C3, D1, C3, C2
Module 5	1-5	B2, B2, B3, C2, C2
	6-10	C2, B2, C3, B3, D1
	11-15	C1, B1, C2, C2, B2
	16-20	C2, C2, C1, C1, C3

			DIFFICULTY LEVEL		
			1	2	3
			LOW	MEDIUM	HIGH
Learning Objectives	A	Remember	A1	A2	A3
	B	Understand	B1	B2	B3
	C	Apply	C1	C2	C3
	D	Analyze	D1	D2	D3

Module 1
Complex Variable - Differentiation

1 Complex functions

Definition 1.1. *Let S be a set of complex numbers. A complex function on S is a rule that assigns to every z in S a complex number w, called the value of f at z. We write this as $w = f(z)$. z varies in S and is called a complex variable. S is called the domain of f.*

Result :

Every complex function can be written as $w = f(z) = u(x, y) + iv(x, y)$

2 Limit, continuity and differentiation

Definition 2.1. *A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined, and $\lim_{z \to z_0} f(z) = f(z_0)$. A function f is continuous in a domain if it is continuous at each point in the domain.*

3 Analytic functions

Definition 3.1. *A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. $f(z)$ is said to be analytic* *at a point $z = z_0$ in D if $f(z)$ is analytic in a small open disc containing z_0. Another term for analytic functions is holomorphic function.*

4 Cauchy - Riemann equations

Theorem 4.1. *Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point z and differentiable at z itself. Then u and v satisfy the Cauchy - Riemann equations $u_x = v_y, \quad u_y = -v_x$ at that point.*

Remark 1. *Hence, if $f(z)$ is analytic on a domain D, these partial derivatives exist and satisfy the Cauchy - Riemann equations at all points of D .*

5 Laplace's equation and Harmonic functions

Laplace's equation

$$\nabla^2 z = z_{xx} + z_{yy} = 0$$

Theorem 5.1. *If $f(z) = u(x, y) + iv(x, y)$ is analytic on a domain D, then both u and v satisfy the Laplace's equation.*

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \ \nabla^2 v = v_{xx} + v_{yy} = 0$$

Definition 5.1. *Functions which satisfy Laplace's equation are called harmonic functions.*

Definition 5.2. *If u and v are harmonic functions such that $f = u + iv$ is analytic, then v is called the harmonic conjugate of u.*

5.1 Finding the harmonic conjugate

Given a function u, we are interested in finding its harmonic conjugate. The steps involved in this calculation are:

1 Check that u is harmonic, that is, it satisfies the Laplace's equation.

2 Let the harmonic conjugate of u be v. Then $f = u + iv$ is analytic

3 u and v should satisfy the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$

4 Integrate v_x with respect to x (or v_y with respect to y) to find $v(x, y)$. This v will include an arbitrary function of y (arbitrary function of x, if you had integrated with respect to y).

5 Now use the remaining Cauchy-Riemann equation to find the arbitrary function

6 Conformal Mapping

A complex function $w = f(z) = u(x, y)+iv(x, y)$ of a complex variable $z = x+iy$ gives a mapping of its domain D in the complex z plane into the complex w plane. For any point z_0 in D, the point $w_0 = f(z_0)$ is called image of z_0 with respect to f. Conformal mappings are those mappings that preserve angles, except at critical points and these mappings are defined by analytic function.

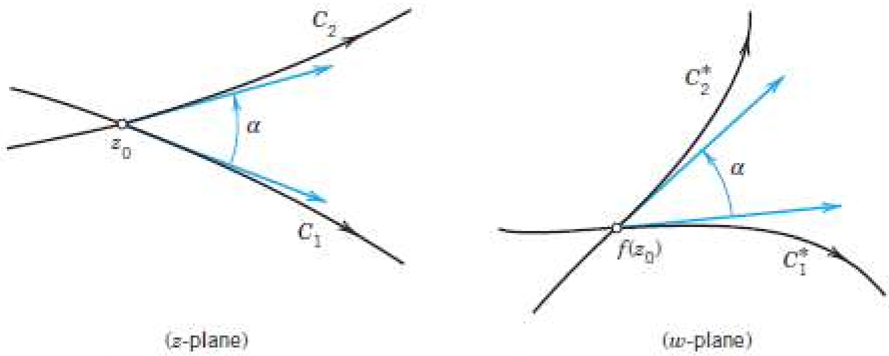


Fig. 380. Curves C_1 and C_2 and their respective images C_1^* and C_2^* under a conformal mapping $w = f(z)$

7 Linear Fractional Transformation or Mobius transformation

A transformation $w = \dfrac{az + b}{cz + d}$ where a, b, c and d are constants and $ad - bc \neq 0$ is called linear fractional transformation.

NOTE; $w' = \frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$, $ad-bc \neq 0 \Rightarrow$ conformal for all z

The following standard transformation are special case of linear fractional transformation

1 Translation:$w = z + a$

2 Rotation: $w = az, \ |a| = 1$

3 Inversion: $w = \frac{1}{z}$ (inversion in the unit circle)

4 Linear Transformation: $w = az + b$

Definition 7.1. *Fixed points of a mapping $w = f(z)$ are points that are mapped onto themselves(kept fixed) i.e. $w = f(z) = z$*

Transformation	Number of Fixed points
Identity map $w = z$	Every points
$w = \bar{z}$	Infinitely many
Inversion $w = \frac{1}{z}$	Two
Rotation $w = cz$	One
Translation $w = z + c$	None

TUTORIAL

1 Show that $f(z) = \begin{cases} \frac{(x+y)^2}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is discontinuous at $z = 0$.

2 Show that $f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not differentiable at $z = 0$.

3 Find an analytic function whose real part is $e^x \cos y$.

4 Find the points at which the function $w = cos\ z$ is not conformal

5 Find the image of the circle $x^2 + y^2 + 4y = 0$ under the transformation $w = \frac{1}{z}$

ASSIGNMENT

1 Show that $\lim_{z\to 0} \frac{x^2y}{x^4+y^2}$ does not exist even though this function approaches the same limit along every straight line through the origin.

2 Find out, and give reason, whether

$$f(z) = \begin{cases} (Re\ z^2)/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous at $z = 0$.

3 Show that $w = \sin z$ is analytic everywhere. Also find its derivative.

4 Show that

$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at $z = 0$, but not differentiable at $z = 0$.

5 Find the value of a so that $u = xy + ax^2 - y^2$ is harmonic. Find its harmonic conjugate.

6 Find the harmonic conjugate of $u = \frac{x}{x^2+y^2}$

7 Find the critical point and fixed point of $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

8 Discuss the transformation $w = cos\ z$

9 Find all linear fractional transformation with fixed points $z = 0$

10 (a) Give an example of a harmonic function which is not analytic

(b) Give an example linear fractional transformation which have no fixed points.

UNIT-WISE QUESTION BANK

1 Show that the function

$$f(z) = \begin{cases} (\frac{2xy^2}{x^2+3y^4} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is discontinuous at $z = 0$.

2 Show that the function $f(z) = \sinh z$ is analytic everywhere. Also find its derivative.

3 Show that the function $f(z) = \sqrt{x+y}$ is not differentiable at $z = 0$.

4 Prove that the function

$$f(z) = \begin{cases} \frac{x^2y^5(x+iy)}{x^4+y^{10}} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is not differentiable at origin, although Cauchy-Riemann equations are satisfied at the origin.

5 Prove that $u = x^2 + y^2$ is not a harmonic function.

6 Find an analytic function whose imaginary part is $\frac{x-y}{x^2+y^2}$.

7 Find a so that the function $u = x^3 + axy^2$ is harmonic. Find its harmonic conjugate.

8 If u is harmonic prove that $u_x - iu_y$ is analytic.

9 Find all points at which the function $w = cosh\ 2\pi z$ is not conformal

10 Find all linear fractional transformation with fixed points $z = \pm i$

11 Find all linear fractional transformation with fixed points $z = \pm 1$

12 Find the fixed points of the transformation $w = \dfrac{iz+4}{2z-5i}$

13 Find the fixed points of the transformation $w = \dfrac{aiz-1}{z+ai}$

14 Find the inverse of the inverse of the transformation $w = \dfrac{i}{2z-1}$

15 Find the inverse of the inverse of the transformation $w = \dfrac{z - \frac{i}{2}}{\frac{-iz}{2} - 1}$

16 Prove that an analytic function with a constant modulus is constant.

17 Prove that an analytic function with a constant real part is a constant.

18 Is the function $u = 2xy + 3xy^2 - 2y^3$ harmonic ?

19 Find the harmonic conjugate of $u = \dfrac{x}{x^2 + y^2}$

20 Prove that $w = cos\ hz$ is an entire function. Also find its derivative.

MODULE 2 - COMPLEX INTEGRATION

READY RECKONER

1 Cauchy's Integral Theorem

Definition 1.1. *A simple closed path is a closed path that does not intersect or touch itself. For example, a circle is simple but a curve shaped like an 8 is not simple.*

Definition 1.2. *A simply connected domain D in the complex plain is a domain such that every simple closed path in D encloses only points of D.*

1.1 Cauchy's Integral Theorem

Statement

If $f(z)$ is analytic in a simple connected domain D, then for every simple closed path C in D, $\oint_C f(z)\,dz = 0$.

Example 1. *For any closed path C, since these functions are analytic $\forall$ z*

(i) $\oint_C e^z\,dz = 0$

(ii) $\oint_C cos\ z\,dz = 0$

(iii) $\oint_C z^n\,dz = 0\ n = 0, 1, ...$

(iv) $\int_C \frac{1}{2z-1} \neq 0\quad |c| : |z| = 1$

(v) $\int_C \bar{z}\ dz = \int_0^{2\pi} e^{-it}ie^{it}dt = 2\pi i \neq 0$ *where* $C : z(t) = e^{it}$ *is a unit circle.*

2 Cauchy's Integral Formula

Statement 1.

If $f(z)$ is analytic in a simple connected domain D, then for any point z_0 in D and any simple closed path C in D that encloses z_0,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

, the integration being taken counterclockwise.

Definition *Statement 2*

If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D which are then also analytic functions in D. The values of these derivatives at a point z_0 in D are given by the formula

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

and in general

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D.

Example 2. *Evaluate* $\oint_C \frac{Cos\ \pi z}{(z-\pi i)^2}dz$ *for any contour C enclosing the point πi.*

Solution *By Cauchy's Integral Formula,*

$$\oint_C \frac{Cos\ \pi z}{(z-\pi i)^2}dz = 2\pi i(cos\ z)'\Big|_{\pi i} = -2\pi i sin\ \pi i = 2\pi\ sinh\ \pi.$$

Example 3. *For any contour enclosing the point $-i$, find* $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz$

Solution By Cauchy's Integral formula,

$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz = \frac{2\pi i}{2!}(z^4-3z^2+6)''\Big|_{z=-i} = \pi i[12z^2-6]_{z=-i} = -18\pi i$$

Example 4. *For any contour C, 1 lies inside and $\pm 2i$ lie outside, Evaluate* $\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz$

Solution By Cauchy's Integral formula,

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz = \oint_C \frac{\frac{e^z}{z^2+4}}{(z-1)^2} = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1} = 2\pi i\frac{e^z(z^2+4)-e^z 2z}{(z^2+4)^2}\Big|_{z=1} = \frac{6\pi e}{25}i$$

Example 5. *Evaluate* $\oint_C \frac{sin\ z}{z^4}dz$ *where* $|C| : |z| = 1$

Solution By Cauchy's Integral formula,

$$\oint_C \frac{sin\ z}{z^4}dz = \frac{2\pi i}{3!}(sin\ z)'''\Big|_{z=0} = \frac{\pi i}{3}(-cos\ z)\Big|_{z=0} = -\frac{\pi}{3}$$

3 Taylor and Maclaurin series

Definition *The Taylor series of a function $f(z)$ centered at z_0 is given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

If $z_0 = 0$, this series is called the Maclaurin series.

Important Special Taylor Series

(i) **Geometric Series**:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad \text{converges when } |z| < 1$$

(ii) **Binomial series**:

$$\frac{1}{(1+z)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n = 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

converges when $|z| < 1$

(iii) **Exponential function**:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

(iv) **Trigonometric functions**:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

(v) **Hyperbolic functions**:

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

(vi) **Logarithm**:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

In practice, evaluating the Taylor series of a function directly using the definition is difficult. So we often use known Taylor series to compute the Taylor series for a given function

Example 6. *Find the Maclaurin series of* $f(z) = \dfrac{1}{1+z^2}$.

The given function is similar to the sum of the geometric series.

$$f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = 1 - z^2 + z^4 - z^6 + \cdots$$

As the geometric series converges when its common ratio has magnitude less than 1, the above series converges when $|z^2| < 1$, *that is, when* $|z| < 1$.

Example 7. *Find the Maclaurin series of* $f(z) = \tan^{-1} z$.

If we differentiate the given function, from the previous question, we have

$$f'(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$

Now integrate both sides of the above equation

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

Example 8. *Find the Taylor series of* $\dfrac{1}{c-z}$ *centered at* z_0.

The Taylor series centered at z_0 *will contain powers of* $z - z_0$. *To get those terms, add and subtract* z_0 *in the denominator.*

$$\frac{1}{c-z} = \frac{1}{c - z_0 + z_0 - z} = \frac{1}{(c-z_0)-(z-z_0)}$$

Now use geometric series.

$$= \frac{1}{(c-z_0)\left(1-\frac{z-z_0}{c-z_0}\right)} = \frac{1}{(c-z_0)} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{c-z_0}\right)^n$$

This series converges when $\left|\frac{z-z_0}{c-z_0}\right| < 1$*, that is, when* $|z-z_0| < |c-z_0|$.

Example 9. *Develop the following function in powers of* $z-1$.

$$f(z) = \frac{2z^2+9z+5}{z^3+z^2-8z-12}$$

Decomposing the given function using partial fractions,

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3}$$

To get powers of $z-1$*, add and subtract 1 in both the denominators.*

$$= \frac{1}{(z-1+1+2)^2} + \frac{2}{z-1+1-3}$$

$$= \frac{1}{(3+(z-1))^2} - \frac{2}{2-(z-1)}$$

Now use binomial and geometric series.

$$= \frac{1}{9}\left[\frac{1}{\left(1+\frac{z-1}{3}\right)^2}\right] - \frac{1}{1-\left(\frac{z-1}{2}\right)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$$

The first series in the above sum converges when $\left|\frac{z-1}{3}\right| < 1$*, or* $|z-1| < 3$.

The second series in the above sum converges when $\left|\frac{z-1}{2}\right| < 1$*, or* $|z-1| < 2$.

Hence the sum of the above 2 series converges in the region where both the series converge. This region is $|z-1| < 2$.

4 Laurent's Series

Laurent series generalize Taylor series. If we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at z_0 (we cannot use a Taylor series), then we can use a new kind of series called **Laurent's Series**.

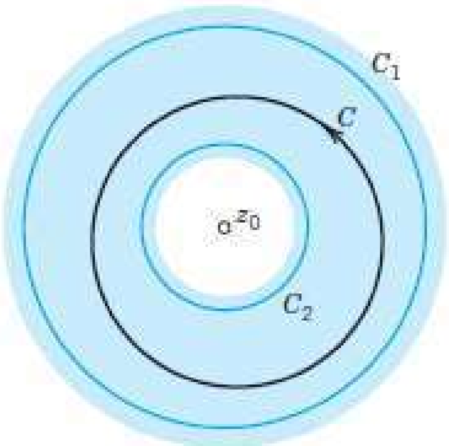
Let $f(z)$ be analytic in a domain containing two concentric circles and with center z_0 and the annulus between them. Then $f(z)$ can be represented by the Laurent series

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n} \\ &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ... + b_0 + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + ... \end{aligned}$$

consisting of non-negative and negative powers. The co-efficient of Laurent's series can be defined are given by the integrals

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz, \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{-(n+1)}} dz^*$$

taken counterclockwise around any closed path C that lies in the anuulus and encircles in inner circle.



In this case the series (or finite sum) of the negative powers is called the **principal part** of $f(z)$ at z_0 [or of that Laurent series].

Example 1. *Find the Laurent's series of $z^{-5} \sin z$ with center* 0

$$f(z) = z^{-5} sin\ z = \frac{sin\ z}{z^5}$$

By Maclaurin's series $sin\ z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....\ (|z| > 0)$

$$\begin{aligned} f(z) &= \frac{sin\ z}{z^5} \\ &= \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right] \\ &= \frac{1}{z^4} - \frac{1}{3!\ z^2} + \frac{1}{5!} - \frac{z^2}{7!} + ... \end{aligned}$$

Example 2. *Find the Laurent's series of* $z^2\ e^{\frac{1}{z}}$ *with center* 0

By Maclaurin's series $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\ \frac{1}{z^2} + \frac{1}{3!}\ \frac{1}{z^3} + \frac{1}{4!}\ \frac{1}{z^4} + ...\ (|z| > 0)$

$$\begin{aligned} f(z) &= z^2\ e^{\frac{1}{z}} \\ &= z^2\left[1 + \frac{1}{z} + \frac{1}{2!}\ \frac{1}{z^2} + \frac{1}{3!}\ \frac{1}{z^3} + \frac{1}{4!}\ \frac{1}{z^4} + ...\right] \\ &= z^2 + z + \frac{1}{2!} + \frac{1}{3!\ z} + \frac{1}{4!\ z^2} + ... \end{aligned}$$

Example 3. *Find Laurent series of* $\frac{1}{z^3 - z^4}$ *with center zero*

$$\begin{aligned} f(z) &= \frac{1}{z^3 - z^4} \\ &= \frac{1}{z^3(1-z)} \\ &= \frac{1}{z^3}(1-z)^{-1} \\ &= \frac{1}{z^3}(1 + z + z^2 + z^3 + z^4 + ...)\ (|z| > 0) \\ &= \frac{1}{z^2} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + ... \end{aligned}$$

Example 4. *Find the Laurent's series expansion of* $\frac{e^z}{(z-1)^2}$ *with center* $z_0 = 1$.

Put $z-1=t \Rightarrow\ z=t+1$

$$\begin{aligned}
f(z) &= \frac{e^z}{(z-1)^2} \\
&= \frac{e^{t+1}}{t^2} \\
&= \frac{e^t.e^1}{t^2} \\
&= \frac{e}{t^2}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+....\right]\ (|t|>0) \\
&= e\left[\frac{1}{t^2}+\frac{1}{t}+\frac{1}{2!}+\frac{t}{3!}+\frac{t^2}{4!}+....\right] \\
&= e\left[\frac{1}{(z-1)^2}+\frac{1}{(z-1)}+\frac{1}{2!}+\frac{(z-1)}{3!}+\frac{(z-1)^2}{4!}+....\right]\ (|z-1|>0)
\end{aligned}$$

Example 5. *Using Laurent's series expand* $\frac{1}{z^2-3z+2}$ *in the region*

(i) $|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>1$

(iv) $0<|z-1|<1$

Here $f(z)=\frac{1}{z^2-3z+2}=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$ (By partial fractions)

(i) $|z|<1 \Rightarrow \frac{|z|}{2}<1$

$$\begin{aligned}
f(z) &= \frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z-1} \\
&= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1-z)^{-1} \\
&= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right)+(1+z+z^2+z^3+....) \\
&= \frac{1}{2}+\frac{3z}{4}+\frac{7z^2}{8}+\frac{15z^2}{16}+....
\end{aligned}$$

(ii) $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1$ *and* $\frac{|z|}{2} < 1$

$$\begin{aligned} f(z) &= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\ &= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\ &= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+....\right) \\ &= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right) - \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right) \end{aligned}$$

(iii) $|z| > 1 \Rightarrow 1 < |z| \Rightarrow \frac{1}{|z|} < 1$ (refer previous cases)

(iv) $0 < |z-1| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z-2} - \frac{1}{z-1} \\ &= \frac{1}{(z-1-1)} - \frac{1}{z-1} \\ &= \frac{1}{-(1-(z-1))} - \frac{1}{z-1} \\ &= -(1-(z-1))^{-1} - \frac{1}{z-1} \\ &= -\left(1+(z-1)+(z-1)^2+(z-1)^3+...\right) - \frac{1}{z-1} \end{aligned}$$

5 Singularities and Zeros

Definition 2.1. *A function $f(z)$ is singular or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic at $z = z_0$*
*No other singularities in the neighborhood of singularity z_0 of $f(z)$, then z_0 is called **Isolated singularity**, otherwise it is called non-isolated singularity.*

Example 6. *tan z has an isolated singularity at $z = \pm\frac{\pi}{2}.\pm\frac{3\pi}{2},....$ and tan $\frac{1}{z}$ has an non-isolated singularity at $z = 0$.*

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by Laurent's series

(i) Removable singularity:- No negative power terms in Laurent's series expansion

Example 7. $\frac{sin\ z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ......\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ......$

$\therefore\ \frac{sin\ z}{z}$ *has removable singular point at* $z = 0$

(ii) Isolated essential singularity:-Infinite number of negative power terms in Laurent's series expansion

Example 8. $f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + ...$ *at* $z = 0$ *is an isolated essential singularity.*

(iii) Pole type singularity:-Finite number of negative power terms in Laurent's series expansion

Example 9. *(a)*

$$\begin{aligned} f(z) &= \frac{sin\ z}{z^5} \\ &= \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right] \\ &= \frac{1}{z^4} - \frac{1}{3!\ z^2} + \frac{1}{5!} - \frac{z^2}{7!} + ... \end{aligned}$$

$\therefore\ z = 0$ *is a pole type singularity of order 4*

(b) $f(z) = \frac{1}{z(z-2)^5}$, $z = 0$ *is a simple pole and* $z = 2$ *is a pole of order 5.*

Zeros of Analytic Function

Definition *A zero of an analytic function* $f(z)$ *in a domain* D *is* $z = z_0$ *in* D *such that* $f(z_0) = 0$*. A zero* z_0 *of* $f(z)$ *has order* n *if* $f(z_0) = 0,\ f^{'}(z_0) = 0,\ f^{''}(z_0) = 0, ..., f^{n-1}(z_0) = 0$ *and* $f^n(z) \neq 0$*.*

Example 10. $f(z) = sin(z^2),\ z = 0$ *is a zero at* $z - 0$

$f(z) = sin(z^2)$ $\qquad f(0) = 0$

$f^{'}(z) = cos(z^2)(2z)$ $\qquad f^{'}(0) = 0$

$f^{''}(z) = 2(cos(z^2) - zsin(z^2).2z)$ $\qquad f^{''}(0) = 2 \neq 0$

$\therefore$, $f(z)$ *has zero at* $z = 0$ *of order 2.*

Example 11. $f(z) = 1 - cos\ z$ *has second order zeros at* $z = 0,\ \pm 2\pi,\ \pm 4\pi,\ ...$

6 Residue Integration

Some real integrals appearing in applications cannot be evaluated easily using usual methods. Complex integration is useful because it allows us to evaluate such real integrals.

The method we will use to evaluate such integrals is called residue integration.

Let $f(z)$ be singular at z_0 but otherwise analytic. Then $f(z)$ has a Laurent series centered at z_0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

Definition *The term b_1 in the Laurent series expansion of $f(z)$ is called the Residue of $f(z)$ at $z = z_0$.*

It is denoted by $b_1 = Res_{z=z_0} f(z)$

Let $f(z)$ be singular at z_0 but otherwise analytic. Let C be a simple closed path that contains $z = z_0$ in its interior, but no other singular points.
Then

$$\oint_C f(z)dz = 2\pi i b_1 = 2\pi i Res_{z=z_0} f(z)$$

where the integral is taken anticlockwise

Example 12. *Evaluate $\oint_{|z|=1/2} e^{1/z} dz$.*
Solution:

(i) *The function to be integrated $f(z) = e^{1/z}$ has a singularity at $z = 0$.*

(ii) *$z = 0$ is inside the given contour, so we can apply the above result.*

(iii) *The Laurent series of $e^{1/z}$ centered at $z = 0$ is*

$$e^{1/z} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots$$

(iv) *$Res_{z=0} e^{1/z} = \frac{1}{1!} = 1$*

(v) *$\oint_{|z|=1/2} e^{1/z} dz = 2\pi i Res_{z=0} e^{1/z} = 2\pi i$*

Example 13. *Integrate* $f(z) = \frac{\sin z}{z^4}$ *anticlockwise around the unit circle.*
Solution:

(i) $f(z)$ *has a singularity at* $z = 0$.

(ii) $z = 1$ *is inside the contour* $|z| = 1$.

(iii) *The Laurent series of* $f(z)$ *centered at* $z = 0$ *is*

$$\frac{\sin z}{z^4} = \frac{1}{z^4} - \frac{1}{3!z} + \frac{z^2}{5!} - \cdots$$

(iv) $Res_{z=0}f(z) = -\frac{1}{3!}$

(v) $\oint_{|z|=1} \frac{\sin z}{z^4} dz = 2\pi i Res_{z=0} \frac{\sin z}{z^4} = -2\pi i \frac{1}{3!} = -\frac{\pi i}{3}$

7 Formula for Residues

The residue of a function $f(z)$ at its poles can be found using the below formulas:

Residue at a simple pole
We have 2 methods for evaluating the residue:

(i) $Res_{z=z_0}f(z) = \lim_{z\to z_0}(z - z_0)f(z)$

(ii) If $f(z) = p(z)/q(z)$, $p(z_0) \neq 0, q(z_0) = 0$ and q has a simple pole at z_0, then f has a simple pole at z_0.
In this case, $Res_{z=z_0}f(z) = \frac{p(z_0)}{q'(z_0)}$

Example 14. *Find the residue at* $z = i$ *of* $f(z) = \frac{9z+i}{z^3+z}$.
Solution: Factorising the denominator, $f(z) = \frac{9z+i}{z^3+z} = \frac{9z+i}{z(z+i)(z-i)}$.

$f(z)$ *has a simple pole at* $z = i$.

Using the first method,

$$Res_{z=i}\frac{9z+i}{z^3+z} = \lim_{z\to i}\frac{(z-i)(9z+i)}{z(z+i)(z-i)} = \lim_{z\to i}\frac{9z+i}{z(z+i)} = -5i$$

Using the second method,

$$Res_{z=i}\frac{9z+i}{z^3+z} = \left(\frac{9z+i}{3z^2+1}\right)_{z=i} = -5i$$

Residue at a pole of order m

If $f(z)$ has a pole of order m at z_0, the residue is $Res_{z=z_0}f(z) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]$

Example 15. *Find* $Res_{z=1}f(z)$, *where* $f(z) = \frac{50z}{z^3+2z^2-7z+4}$

Solution:

Factorising the denominator, $f(z) = \frac{50z}{z^3+2z^2-7z+4} = \frac{50z}{(z+4)(z-1)^2}$.

$f(z)$ *has a pole of order 2 at* $z = 1$.

$$Res_{z=1}f(z) = \lim_{z\to 1}\left[\frac{d}{dz}(z-1)^2 f(z) =\right] = \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{50z}{z+4}\right)\right] = 8$$

8 Cauchy Residue Theorem

Theorem 1. *Let* $f(z)$ *be analytic inside a simple closed path* C *and on* C, *except for finitely many singular points* $z_1, z_2, \cdots, z_n$ *inside* C.

Then the integral of $f(z)$ *taken anticlockwise around* C *equals* $2\pi i$ *times the sum of the residues of* $f(z)$ *at* $z_1, z_2, \cdots, z_n$, *ie,*

$$\oint_C f(z)dz = 2\pi i \Sigma_{j=1}^{n} Res_{z=z_j} f(z)$$

9 Applications of Cauchy Residue Theorem

Evaluation of definite integral using Residue theorem

Example 16. *Evaluate the integral* $\oint_C \frac{4-3z}{z^2-z}$ *anticlockwise around the given path*

(i) $|z| = 2$

(ii) $|z| = 1/2$

(iii) $|z-1| = 1/2$

(iv) $|z-2| = 1/2$

Solution:

The function to be integrated has singularities at $z = 0, 1$.

The residues at these singularities are

$$Res_{z=0}\frac{4-3z}{z(z-1)} = -4, \quad Res_{z=1}\frac{4-3z}{z(z-1)} = 1$$

Now evaluate the integral over each path

(i) *Both singularities are inside the contour ,so by Residue theorem*

$\oint_{|z|=2} \frac{4-3z}{z^2-z} = 2\pi i \left[Res_{z=0}\frac{4-3z}{z(z-1)} + Res_{z=1}\frac{4-3z}{z(z-1)}\right] = -6\pi i.$

(ii) *Only $z = 0$ is inside the contour, so again by Residue theorem*

$\oint_{|z|=1/2} \frac{4-3z}{z^2-z} = 2\pi i \left[Res_{z=0}\frac{4-3z}{z(z-1)}\right] = -8\pi i.$

(iii) *Only $z = 1$ is inside the contour, so again by Residue theorem*

$\oint_{|z-1|=1/2} \frac{4-3z}{z^2-z} = 2\pi i \left[Res_{z=1}\frac{4-3z}{z(z-1)}\right] = 2\pi i.$

(iv) *Neither of the singularities are inside the contour, so by Cauchy's theorem*

$\oint_{|z-2|=1/2} \frac{4-3z}{z^2-z} = 0.$

Example 17. *Evaluate the integral* $\oint_{|z|=3/2} \frac{\tan z}{z^2-1}$

Solution:

Here, we can rewrite the function to be integrated as $\frac{\tan z}{z^2-1} = \frac{\sin z}{(z+1)(z-1)\cos z}$.

This function has singularities at $z = 1, -1, (2n+1)\frac{\pi}{2}$.

Of these, only $z = 1, -1$ are inside the contour.

Therefore, by Residue theorem

$$\oint_{|z|=3/2} \frac{\tan z}{z^2-1} = 2\pi i \left[Res_{z=1}\frac{\tan z}{z^2-1} + Res_{z=-1}\frac{\tan z}{z^2-1}\right] = 2\pi i \tan(1)$$

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MODULE 2 COMPLEX VARIABLE - INTEGRATION

TUTORIAL

1. Find the Maclaurin series of sin(2Z) .

2. Find the Taylor series of $f(z) = \frac{1}{z}$ with center at $z_0 = i$

3. Evaluate $\int_C Re(z)dz$, where C is the shortest path from $1 + i$ to $5 + 5i$.

4. Evaluate $\oint_C \frac{cos\, h\ (z^2 - \pi\ i)}{z - \pi i} dz$ where c is a boundary of a quadrilateral with vertices $\pm 2,\ \pm 4i$.

5. Indicate whether cauchy's integral theorem applies for the integration of $f(z)$ counter clockwise around the unit circle.

   (a) $f(z) = \frac{1}{4z - 3}$ $(b)\ f(z) = z^3\ cot\ z$

ASSIGNMENT

1. Find the Maclaurin series of $\frac{z+2}{1-z^2}$

2. Find the Taylor series of $f(z) = \frac{1}{1+z}$ with center at $z_0 = -i$.

3. Evaluate $\int_C Re(z)dz$, where C is the parabola $y = 1 + \frac{1}{2}(x-1)^2$ from $1 + i$ to $3 + 3i$.

4. Evaluate $\int_C Im(z^2)dz$ counterclockwise around the triangle with vertices $0, 1, i$.

5. Evaluate $\int_C \sec^2 z dz$, where C is any path from $\frac{\pi}{4}$ to $\frac{\pi i}{4}$

6. Evaluate $\int_C tan\ z dz$ where C is $|z| = 2$.

7. Evaluate $\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^4}\right] dz$ where C is the circle $|z| = 2$.

8. Evaluate $\int_C \frac{e^z}{cos\ \pi z} dz$ where C is $|z| = 1$

9. Evaluate $\int_C \frac{z+4}{z^2 + 2z + 5} dz$ where C is $|z + 1 - i| = 2$

10. Evaluate $\int_C \frac{sin\ (\pi z^2) + cos\ (\pi z^2)}{(z-1)(z-2)} dz$ where C is $|z| = 3$

UNIT-WISE QUESTION BANK

1. Find the Maclaurin series of $\sin^2(z)$
2. Find the Taylor series of $f(z) = \cos z$ with center at $z_0 = \pi$
3. Find the Taylor series of $f(z) = \dfrac{1}{(z-i)^2}$ with center at $z_0 = -i$.
4. Evaluate $\displaystyle\int_C e^z dz$, where C is the shortest path from $\dfrac{\pi}{2i}$ to $\pi i i$.
5. Evaluate $\displaystyle\int_C z e^{z^2} dz$, where C is the path from 1 along the axes to i.
6. Evaluate $\displaystyle\int_C Re(z^2) dz$ clockwise around the boundary of the square with vertices $0, i, 1+i, 1$.
7. Show that $\displaystyle\int_C \frac{1}{z} dz = \pi i$ or $-\pi i$ according as C is the semicircle $|z| = 1$ above or below the real axis from $(1, 0)$ to $(-1, 0)$.
8. Evaluate $\displaystyle\int_C \frac{cos\pi z}{z^2-1} dz$ where C is the rectangle with vertices $2 \pm i, \ -2 \pm i$
9. Evaluate $\displaystyle\int_C \frac{e^z}{(z+1)^3} dz$ where C is $|z+1| = 1$
10. Evaluate $\displaystyle\int_C \frac{z^2+5z+3}{(z-2)^3} dz$ where C is $|z| = 3$
11. Using Cauchy's integral formula, Evaluate $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)} dz$ where C is $|z-2| = 2$
12. Evaluate $\displaystyle\int_C \frac{cos\ \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$
13. Evaluate $\displaystyle\int_C \frac{2z-1}{z(z+1)(z-1)} dz$ where C is $|z| = 2$
14. Evaluate $\displaystyle\int_C \frac{dz}{z^2(z-1)} dz$ where C is $|z| = 2$
15. Evaluate $\displaystyle\int_C \frac{2z-1}{z(z+1)(z-1)} dz$ where C is $|z| = 2$
16. Evaluate $\displaystyle\int_C \frac{ze^z}{(z+1)(z+2)} dz$ where C is $|z-i| = 3$

17· Find the Laurent series expnasion of $\dfrac{cos\ z}{z^4}$ at the singular point.

18. Find the Laurent series expnasion of $\dfrac{sinh\ 2z}{z^2}$ at the singular point $z = 0$.

19. Find the Laurent series expnasion of $\dfrac{cos\ z}{z^4}$ at the singular point.

20. Find the Laurent series expnasion of $\dfrac{sinh\ 2z}{z^2}$ at the singular point $z = 0$.

MODULE 3 -Laplace Transforms

Ready Reckoner

- If $f(t)$ is a function defined for all $t \geq 0$, its **Laplace transform** is given by:

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t)dt$$

- **Inverse Laplace transforms:**

$$\mathcal{L}[f(t)] = F(s) \Leftrightarrow \mathcal{L}^{-1}F(s) = f(t)$$

- **Linearity of Laplace transforms:**

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$$

- **Laplace transform of basic functions:**

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$
$t^n (n = 0,1, \dots)$	$\frac{n!}{s^{n+1}}$
$t^a (a\ positive)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
e^{at}	$\frac{1}{s-a}$
$cos\ wt$	$\frac{s}{s^2 + w^2}$

$sin\ wt$	$\frac{w}{s^2+w^2}$
$cosh\ at$	$\frac{s}{s^2-a^2}$
$sinh\ at$	$\frac{a}{s^2-a^2}$
$e^{at}coswt$	$\frac{s-a}{(s-a)^2+w^2}$
$e^{at}sinwt$	$\frac{w}{(s-a)^2+w^2}$

- **First Shifting Theorem:** $\mathcal{L}[e^{at}f(t)] = F(s-a)$ or $e^{at}f(t) = \mathcal{L}^{-1}[F(s-a)]$
- **Laplace transform of Derivatives and Integrals:**

$$\mathcal{L}[f'] = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}[f''] = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

In general, $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

Also, $\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$

thus, $\int_0^t f(\tau)d\tau = \mathcal{L} - 1\left[\frac{1}{s}F(s)\right]$

- **Multiplication by t^n**

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

- **Solving ODEs using Laplace Transforms:**

Step 1: Apply Laplace transforms to the given differential equation

$$y'' + ay' + by = r(t)$$

Step 2: Writing $\mathcal{L}(y) = Y(s)$, we obtain the **subsidiary equation**

$$(s^2 + as + b)Y = \mathcal{L}(r) + sf(0) + f'(0) + af(0)$$

Step 3: Solve the subsidiary equation algebraically for $Y(s)$.

Step 4: Determine the inverse Laplace transform $y(t) = \mathcal{L}^{-1}[Y(s)]$, which is the solution to the ODE.

- **Unit Step Function (Heaviside Function):**

$$u(t-a)=\begin{cases}0\,, t<a\\1\,, t>a\end{cases}$$

$$\mathcal{L}[u(t-a)]=\frac{e^{-as}}{s}$$

- **Second Shifting Theorem:**

  If $\mathcal{L}[f(t)]=F(s)$, then

$$\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s)$$

or

$$f(t-a)u(t-a)=\mathcal{L}^{-1}[e^{-as}F(s)]$$

- **Dirac Delta Function:**

$$\delta(t-a)=\begin{cases}0\,, t\neq a\\\infty\,, t=a\end{cases}$$

$$\mathcal{L}[\delta(t-a)]=e^{-as}$$

- **Convolution:**

$$\boldsymbol{h}(\boldsymbol{t})=(\boldsymbol{f}*\boldsymbol{g})(\boldsymbol{t})=\int_{\boldsymbol{0}}^{\boldsymbol{t}}\boldsymbol{f}(\boldsymbol{\tau})\boldsymbol{g}(\boldsymbol{t}-\boldsymbol{\tau})\boldsymbol{d}\tau$$

- **Convolution Theorem:** If two functions f and g satisfy the assumptions of the existence theorem, so that their transforms F and G exist, the product $H=FG$ is the transform of $h=f*g$.

TUTORIAL QUESTIONS

1. Find the Laplace transform of the following functions:

   (i) $f(t) = sinat.sinbt$ (ii) f(t) = $e^{-2t}\cos 4$t (iii) $f(t) = sin^2 3t$

   (iv) $f(t) = e^{-2t} cos\, 5\, t$ (v) $f(t) = t^2$ (vi) $f(t) = (a - bt)^2$

2. Find the Laplace transform of the following functions:

   (i) $f(t) = tcos4t$ (ii) $f(t) = t^3 e^{-2t}$ (iii) $f(t) = t^2 sint$

3. Find the Laplace transform of the following functions:

   (i) $t^2 + te^t + \sin 2t$ (ii) $t - 3, t > 3$ (iii) $t^2, 1 < t < 2$

   (iv) $\sin ht, 0 < t < 2$ (v) $\sin \pi t, 2 < t < 4$

4. Find the Laplace transform of $tf(t)$, where $f(t) = e^{2t} + sin^2 t$

5. Find the Inverse Laplace transform of i) $\frac{s+2}{s^2-4s+13}$ ii) $\frac{s+1}{s^2+2s}$ iii) $\frac{e^{-2s}}{(s-1)^3}$

ASSIGNMENT QUESTIONS

1. Find the Laplace transform of i) $f(t) = coshatsinat$ ii) $\cos(\omega t + \theta)$

2. Find the Laplace transform of $f(t) = ke^{-at}\cos \omega t$

3. Find the Laplace transform of i) $f(t) = cos^2 2t$ ii) $f(t) = t^2 e^{-3t} \sin 2t$

4. Find the Laplace transform of i) $f(t) = te^{at} \sin bt$ ii) $e^{-t} sinh4t$

5. Find the Laplace transform of $f(t) = e^{-2t} \int_0^t tcos3t\, dt$.

6. Find the Laplace transform of i) $f(t) = t^2 \sin at$ ii) $\cos 2t, (0 < t < \pi)$

7. Find the Inverse Laplace transform of i) $\frac{s+2}{s^2(s+1)(s+3)}$ ii) $\frac{4s+12}{s^2+8s+16}$ iii) $\frac{2s+3}{(s^2+2s+5)(s-1)}$

   iv) $\frac{4}{s^2-2s-3}$ v) $\frac{e^{-2s}}{s^6}$ vi) $\frac{2(e^{-s}-e^{-3s})}{(s^2-4)}$

8. Solve the differential equations $(D^2 + \omega^2)x = \cos \omega t, t \geq 0, x(0) = 0, x'(0) = 0.$

9. Solve the differential equations i) $y'' - 3y' + 2y = 4t + e^{3t}; y(0) = 1, y'(0) = -1.$

   ii) $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), y(0) = 1, y'(0) = 1$

10. Apply convolution theorem to evaluate $L^{-1}(\frac{s}{(s^2+a^2)^2})$

UNITWISE QUESTION BANK

1. Find the Laplace transform of i) $f(t) = cos2\pi t$ ii) $f(t) = 2t + 8$

2. Find the Laplace transform of i) $f(t) = 1.5\sin\left(3t - \frac{\pi}{2}\right)$ ii) $cos^2\omega t$

3. Find the Laplace transform of i) $f(t) = sin^2 2t$ ii) $f(t) = te^{-4t}\cos 2t$

4. Find the Laplace transform of i) $f(t) = te^{5t}\sin 7t$ ii) $e^{-3t}sinh6t$

5. Find the Inverse Laplace transform of i) $\frac{5s+1}{s^2-25}$ ii) $\frac{s}{L^2s^2+\frac{1}{4}\pi^2}$ iii) $log\frac{s+1}{s-1}$

6. Find the Inverse Laplace transform of $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

7. Find the Inverse Laplace transform of i) $\frac{90}{(s+\sqrt{3})^6}$ ii) $\frac{6s+7}{2s^2+4s+10}$

8. Find the Inverse Laplace transform of i) $\frac{k_0}{s} + \frac{k_1}{(s-a)^2}$ ii) $\frac{a(s+k)+b\pi}{(s+k)^2+\pi^2}$

9. Solve the differential equations i) $y'' + y' - 6y = 0; y(0) = 1, y'(0) = 1.$

   ii) $y'' + 9y = 10e^{-t}; y(0) = 0, y'(0) = 0.$

10.Solve the differential equation $y' - 6y = 0\,; y(-1) = 4.$

11. Find the Inverse Laplace transform of $\frac{1}{s(s^2+\frac{\omega^2}{4})}$

12. Find the Inverse Laplace transform of i) $\frac{1}{s^3+as^2}$ ii) $\frac{3s+4}{s^4+k^2s^2}$

13. Find the Inverse Laplace transform of $\frac{4(1-e^{-\pi s})}{s^3+4}$

14. Apply convolution theorem to evaluate $L^{-1}(\frac{9}{s(s+3)})$

15. Apply convolution theorem to evaluate $L^{-1}(\frac{e^{-as}}{s(s-2)})$

16. Solve the differential equation $y'' - 3y' + 2y = 4t - 8; y(0) = 0, y'(0) = 3.$

17. Solve the differential equation $y'' + 4y' + 5y = \delta(t-1); y(0) = 2, y'(0) = 7.$

18. Find the Inverse Laplace transform of $F(s) = \frac{-s+11}{s^2-2s-3}$.

19. Find the Inverse Laplace transform of $F(s) = \frac{2}{s^2+\frac{s}{3}}$.

20. Solve the differential equation $y'' + 10y' + 24y = 144t^2; y(0) = \frac{19}{12}, y'(0) = -5.$

Module 4- Fourier Transforms

Ready Reckoner

- **Fourier Integral Representation:**

$$f(x) = \int_0^{\infty} [A(w)coswx + B(w)sinwx]dx$$

where, $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\,(v)cos\;wv\;dv$ and $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\,(v)sin\;wv\;dv$

- **Fourier Cosine Transform:** The Fourier cosine transform concerns even functions.

The Fourier cosine transform of f is given by: $\hat{f_c}(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)cos\;wx\;dx$

and $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f_c}(w)cos\;wx\;dw$

- **Fourier Sine Transform:** The Fourier sine transform concerns odd functions.

The Fourier sine transform of f is given by: $\hat{f_s}(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)sin\,wx\;dx$

and $f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f_s}(w)sin\;wx\;dw$

- **Fourier Transform:** Fourier transform of a function f is given by:

$$\boldsymbol{\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx}$$

- **Inverse Fourier Transform:** $\boldsymbol{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw}$

- **Linearity of Fourier Transform:**

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

where f and g are functions whose Fourier transforms exist and a, b are constants.

- **Fourier Transforms of Derivatives:** $\boldsymbol{\mathcal{F}[f'(x)] = iw\mathcal{F}[f(x)]}$

MODULE 4 - Fourier Transform
Tutorial

1. Use Fourier integral to show that

$$\int_0^\infty \frac{\cos(xw) + w\sin(xw)}{1+w^2} dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

2. Using Fourier sine integral for $f(x) = e^{-ax}$, show that

$$\int_0^\infty \frac{\lambda \sin(\lambda x)}{\lambda^2 + a^2} d\lambda = \pi e^{-ax}$$

3. Find the Fourier sine transform of $e^{-x}, x > 0$. Hence evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{1+x^2} dx$.

4. Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin(sx)dx = e^{-s}$.

5. Find the Fourier transform of

$$f(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

and also find the inverse Fourier transform.

Assignment

1. Using Fourier integral representation, show that

$$\int_0^\infty \frac{\sin(w) - w\cos(w)}{w^2} \sin(xw)dw = \begin{cases} \frac{\pi}{2}x & \text{if } 0 < x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

2. Using Fourier integral, prove that

$$\int_0^\infty \frac{\sin(\pi\lambda)\sin(x\lambda)}{1-\lambda^2} = \begin{cases} \frac{\pi}{2}\sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x \geq \pi \end{cases}$$

3. Using Fourier cosine integral, show that

$$\int_0^\infty \frac{\cos xw}{1+w^2} dw = \frac{\pi}{2} e^{-x} \text{ if } x > 0$$

4. Represent

$$f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

as a Fourier cosine integral.

5. Find the Fourier sine transform of

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

6. Find the Fourier cosine transform of $f(x) = \sin x, 0 < x < \pi$

7. Solve the integral equation

$$\int_0^\infty f(x)\cos(px)dx = \begin{cases} 1-p & 0 \le p \le 1 \\ 0 & p > 1 \end{cases}$$

Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$.

8. Find $f(x)$ from

$$\int_0^\infty f(x)\sin(xt)dt = \begin{cases} 1 & 0 \le t < 1 \\ 2 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

9. Find the Fourier transform of

$$f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

when $k > 0$.

10. Find the Fourier transform of

$$f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

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Unitwise Question Bank

1. Find the Fourier integral representation of

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

2. Show that the integral represents the indicated function

$$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}dw = \frac{1}{2}\pi e^{-x}\cos x \text{ if } x > 0$$

3. Show that

$$\int_0^\infty \frac{\cos(\frac{\pi}{2}w)\cos(xw)}{1-w^2}dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if } 0 < |x| < \frac{\pi}{2} \\ 0 & \text{if } |x| > \frac{\pi}{2} \end{cases}$$

4. Express

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

as a Fourier sine integral. Hence evaluate $\displaystyle\int_0^\infty \frac{1-\cos(\pi w)}{w}\sin(xw)dw$.

5. Represent the following function as a Fourier cosine integral.

$$f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

6. Represent $f(x)$ as a Fourier sine integral.

$$f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

7. Find the Fourier cosine transform of $2e^{-2x} + 3e^{-4x}$.

8. Find $f(x)$, if its sine transform is $\dfrac{e^{-as}}{s}$. Hence deduce the inverse sine transform of $\dfrac{1}{s}$.

9. Find the Fourier cosine transform of

$$f(x) = \begin{cases} 3 & \text{if } 0 \le x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

10. Find $f(x)$ from

$$\int_0^\infty f(x)\cos(xt)dt = \begin{cases} 2 & 0 \le t < 1 \\ 3 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$

11. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$$

12. Find the Fourier transform of

$$f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

13. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

14. Find the Fourier transform of

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

15. Find the Fourier transform of

$$f(x) = \begin{cases} e^{2ix} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

16. Using Fourier integral prove that $\int_0^\infty \frac{\cos\frac{1}{2}\pi\omega}{1-\omega^2}\cos x\omega\, d\omega = \begin{cases} \frac{\pi}{2}\cos x & 0 \le x < \pi \\ 0 & \text{elsewhere} \end{cases}$

17. Find the Fourier transform of $f(x) = \begin{cases} xe^{-x} & 0 \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$.

18. Find $f(x)$, if its sine transform is $\frac{e^{-a\omega}}{\omega}$.

19. Represent $f(x) = \begin{cases} e^{-x} & 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$ as cosine integral.

20. Find the Fourier transform of $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$. Deduce that that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

MODULE 5: Z-Transforms

Ready Reckoner

Z-Transforms

The Z transforms of a sequence $\{u_n\}$ defined for discrete values $u = 0,1,2, \ldots$ and $u_n = 0$, for $n < 0$ is defined by $Z(u_n) = \sum_0^\infty u_n z^{-n}$for all z.

Z-Transforms of some basic functions

(1) $z(1) = \frac{z}{z-1}$

(2) $z(-1)^n = \frac{z}{z+1}$

(3) $z(a^n) = \frac{z}{z-a}$

(4) $z(n) = \frac{z}{(z-1)^2}$

(5) $z(na^n) = \frac{az}{(z-a)^2}$

(6) $z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}$

(7) $z\left(\frac{1}{n+1}\right) = -zln\left(1 - \frac{1}{z}\right)$

Properties of Z-Transform

I. Linearity Property

If a, b, c are constants and u_n, v_n, w_n be any discrete functions, then $Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$

II. Damping Rule

If $Z(u_n) = U(z)$, then $z(a^{-n}u_n) = U(az)$

III. Shifting U_n to the right

If $Z(u_n) = U(z)$, then $z(u_{n-k}) = z^{-k}U(z)$

Initial value Theorem

If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{\text{Lt}}\, U(z)$.

Final value Theorem

If $Z(u_n) = U(z)$, then $\underset{n\to\infty}{\text{Lt}}\, u_n = \underset{z\to 1}{\text{Lt}}\,(z-1)U(z)$.

Inverse Z Transforms

The inverse Z-transform is a process of determining the sequence which generates a given Z-transform.

Convolution Theorem

If $Z^{-1}(U(z)) = u_n$ and $Z^{-1}(V(z)) = v_n$, then $Z^{-1}[U(z)V(z)] = \sum_{m=0}^{n} u_m . v_{n-m} = u_n * v_n$

where the '$*$' denote the convolution operation.

Application to Difference Equation

Method of Z-transform is used to solve difference equation. Also, it plays an important role in the analysis and representation of discrete time system.

TUTORIAL QUESTIONS

1. Find the Z-transform of $(n+1)^2$.
2. Find the Z-transform of $\sin(3n+5)$.
3. Apply damping rule to find the Z-transform of $(i) na^n (ii) n^2 a^n$
4. Find the inverse Z-transform of $\frac{2z^2+3z}{(z+2)(z-4)}$.
5. Use convolution theorem to evaluate the inverse Z-transform of $\frac{z^2}{(z-1)(z-3)}$.

ASSIGNMENT QUESTIONS

1. Find the Z-transform of n^4.
2. Find the Z-transform of $(i) e^{an} (ii) ne^{an} (iii) n^2 e^{an}$.
3. Find the Z-transform of $(i) \cosh n\theta \ (ii) a^n \cosh n\theta$.
4. Find the inverse Z-transform of $\frac{z^3+3z}{(z-1)^2(z^2+1)}$.
5. Find the inverse Z-transform of $\frac{2z}{(z-1)(z^2+1)}$.
6. Solve $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
7. Solve $u_{n+2} - 5u_{n+1} + 6u_n = 5^n$.
8. Solve $u_{n+2} - 2u_{n+1} + u_n = 2^n$, with $u_0 = 2, u_1 = 1$.
9. Solve $u_{n+2} - 5u_{n+1} + 6u_n = y_n$, with $u_0 = 0, u_1 = 1$ and $y_n = 1$ for $n = 0,1,2,3, \ldots$ by Z-transform method.
10. Using Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

UNITWISE QUESTION BANK

1. Find the Z-transform of $(3n - 4\sin\frac{n\pi}{4} + 5a)$, where a is a constant.
2. Find $Z[\sin(3n+5)]$.
3. Find $Z\left(n_{C_p}\right)$.
4. Find the Z-transform of $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$.
5. Show that $Z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}$. Hence evaluate $Z\left(\frac{1}{(n+1)!}\right)$ and $Z\left(\frac{1}{(n+2)!}\right)$.

6. Find the Z-transform of $n \sin n\theta$.

7. Using $Z(n^2) = \frac{z^2+z}{(z-1)^3}$, show that $Z((n+1)^2) = \frac{z^3+z^2}{(z-1)^3}$.

8. Find the Z-transform of $\cos\left(\frac{n\pi}{2}\right)$.

9. Find the Z-transform of $\sin\left(\frac{n\pi}{2}\right)$.

10. Find the Z-transform of $e^{3n} \sin 2n$.

11. Find the invers Z-transform of $\frac{18z^2}{(2z-1)(4z+1)}$.

12. If $U(z) = \frac{2z^2+3z+12}{(z-1)^4}$, find the value of u_2 and u_3.

13. Using linearity property find the Z-transform of $\frac{1}{2}(n+2)(n-1)$.

14. Using linearity property find the Z-transform of $\frac{1}{2}(n+2)^3$.

15. Using convolution theorem find the inverse Z-transform of $\frac{z^2}{(z-1)^2}$.

16. Using convolution theorem find the inverse Z-transform of $\frac{z^2}{(z-4)(z-3)}$.

17. Solve the difference equation $y_{n+3} + y_{n+2} - 8y_{n+1} - 12y_n = 0, y_0 = 1, y_1 = y_2 = 0$.

18. Solve the difference equation $y_{n+3} - 3y_{n+1} + 2y_n = 0, y_1 = 0, y_2 = 8, y_3 = -8$.

19. Solve the difference equation $u_{n+2} + u_n = 5.2^n, u_0 = 1, u_1 = 0$.

20. Find the inverse Z-transform of $\frac{4z^{-1}}{(1-z^{-1})^2}$.